

HÖLDER CONTINUOUS SOLUTIONS OF BOUSSINESQ EQUATION WITH COMPACT SUPPORT

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ABSTRACT. We show the existence of Hölder continuous solution of Boussinesq equations in whole space which has compact support both in space and time.

Keywords: Boussinesq equations, Hölder continuous solution with compact support

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1. INTRODUCTION

In this paper, we consider the following Boussinesq system

$$\begin{cases} v_t + \operatorname{div}(v \otimes v) + \nabla p = \theta e_2, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}, \\ \operatorname{div} v = 0, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}, \\ \theta_t + \operatorname{div}(v\theta) = 0, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \end{cases} \quad (1.1)$$

Here $e_2 = (0, 1)^T$, v is the velocity vector, p is the pressure, θ is a scalar function. The Boussinesq equations arises from many geophysical flows, such as atmospheric fronts and ocean circulations (see, for example, [25],[27]). To understand the turbulence phenomena in fluid mechanics, one needs to go beyond classical solutions. The pair (v, p, θ) on $\mathbb{R}^2 \times \mathbb{R}$ is called a weak solution of (1.1) if they belong to $L^2_{loc}(\mathbb{R}^2 \times \mathbb{R})$ and solve (1.1) in the following sense:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} (\partial_t \varphi \cdot v + v \otimes v : \nabla \varphi + p \operatorname{div} \varphi + \theta e_2 \cdot \varphi) dx dt = 0,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$.

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} (\partial_t \phi \theta + v \cdot \nabla \phi \theta) dx dt = 0,$$

for all $\phi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R})$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} v \cdot \nabla \psi dx dt = 0.$$

for all $\psi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R})$.

The study of weak solutions in fluid dynamics attract more and more peoples interests. One of the famous problem is the Onsager conjecture on Euler equation which says that the incompressible Euler equation admits Hölder continuous weak solution which dissipates kinetic energy. More precisely, the Onsager conjecture on Euler equation can be stated as following:

(1) $C^{0,\alpha}$ solution are energy conservative when $\alpha > \frac{1}{3}$.

(2) For any $\alpha < \frac{1}{3}$, there exist dissipative solutions with $C^{0,\alpha}$ regularity .

The part (1) has been proved by P. Constantin, E. Weinan and E. Titi in [11] and also by P. Constantin, etc. in [5] with slightly weaker assumption.

The part (2) seems more subtle and has been treated by many authors. For weak solutions, the non-uniqueness results have been obtained by V. Scheffer ([28]), A. Shnirelman ([30, 31]) and Camillo De Lellis, László Székelyhidi ([34, 15]). In particular, a great progress in the construction of Hölder continuous solution was made by Camillo De Lellis, László Székelyhidi etc in recent years. In fact, Camillo De Lellis and László Székelyhidi developed an iterative scheme in [17], together with the aid of Beltrami flow on T^3 and Geometric Lemma, and constructed a continuous periodic solution which satisfies the prescribed kinetic energy. The solution is a superposition of infinitely many weakly interacting Beltrami flows. Building on the iterative techniques in [17] and Nash-Moser mollify techniques, they constructed Hölder continuous periodic solutions with exponent $\theta < \frac{1}{10}$, which satisfies the prescribed kinetic energy in [18]. P. Isett in [23] constructed Hölder continuous periodic solutions with any $\theta < \frac{1}{5}$, and the solution has compact support in time. By introducing some new devices in [3], Camillo De Lellis, László Székelyhidi and T. Buckmaster constructed Hölder continuous weak solutions with $\theta < \frac{1}{5}$, which satisfies the prescribed kinetic energy, also see [2]. In R^3 , P. Isett and Sung-jin Oh in [21] constructed Hölder continuous solutions with $\theta < \frac{1}{5}$, which satisfies the prescribed kinetic energy or is a perturbation of smooth Euler flow. Recently, S. Daneri obtained dissipative Hölder solutions for the Cauchy problem of incompressible Euler flow in [12].

Concerning the Onsager conjecture with the critical spatial regularity, namely Hölder exponent $\theta = \frac{1}{3}$, there are also some interesting results. By time localized estimates and careful choice of the parameters in [1], T. Buckmaster constructed Hölder continuous periodic solutions with exponent $\theta < \frac{1}{5}$ in time-space, which for almost every time belongs to C_x^θ , for any $\theta < \frac{1}{3}$ and is compactly temporal supported. Later, by smoothing Reynolds stress for different time intervals using different approach carefully and introducing some novel ideas in [4], Camillo De Lellis, László Székelyhidi and T. Buckmaster constructed Hölder continuous periodic solution which belongs to $L_t^1 C_x^\theta$, for any $\theta < \frac{1}{3}$ and has compact support in time.

Motivated by the above earlier works, we want to know if the similar phenomena can also happen when considering the temperature effects in the incompressible Euler flow which is the Boussinesq system. In [35], we construct continuous solutions for Boussinesq equations on torus which satisfies the prescribed kinetic energy. In this paper, we consider the existence of Hölder continuous solution with compact support both in space and time for Boussinesq equations. The main difficulty is to deal with the interactions between velocity and temperature. Following the general framework of convex integration method developed by DeLellis and Székelyhidi for Euler equations, by establishing the corresponding geometric lemma and constructing oscillatory perturbation which are compatible with Boussinesq equations, we obtained the following results. To state our theorem, we introduce the following set:

For $r > 0$ and $k = (0, 1)^T$ or $(1, 0)^T$, we define $\Theta(r, k)$ as following

$$\Theta(r, k) := \left\{ \theta(t, x) : \theta(t, x) \in C_c^\infty(Q_r; R) \text{ and } \theta(t, x) = \Delta \left(a(t, x) \left(\frac{e^{iNk \cdot x} + e^{-iNk \cdot x}}{N^2} \right) \right) \right\},$$

where $a(t, x) \in C_c^\infty(Q_r; R)$, $Q_r = \{(t, x) \in R^3 : |(t, x)| < r\}$ and $N > 0$.

Theorem 1.1. *For any given positive number r , ε and any given function $\theta_0 \in \Theta(r, k)$, there exist a triple*

$$(v, p, \theta) \in C_c(Q_{2r}; R^2 \times R \times R)$$

such that they solve the system (1.1) in the sense of distribution and

$$v \neq 0, \quad \sup_{t, x} |\theta(t, x) - \theta_0(t, x)| \leq \varepsilon.$$

Moreover, we have

$$v \in C_{t,x}^{\alpha_1}, \quad \theta \in C_{t,x}^{\alpha_2}, \quad p \in C_{t,x}^{\beta}.$$

for any $\beta \in (0, 1)$ and any $\alpha_1 < \frac{1}{28}$, $\alpha_2 < \frac{1}{25}$.

Remark 1.1. In our theorem 1.1, if $\theta = 0$, then it is the Hölder continuous Euler flow with compact support and have been constructed by P. Isett and Sung-jin Oh in [21]. In fact, they construct Hölder continuous Euler flow with Hölder exponent $\frac{1}{5} - \varepsilon$. Moreover, if we take $a(t, x) = 10\varepsilon$ in Q_r and N sufficiently large, we must have $\theta \neq 0$.

Remark 1.2. Similar results also hold for the 3-dimensional Boussinesq system on R^3 with same Hölder exponent.

We briefly give some comments on our proof. In [21], the authors make use of a families of Beltrami flows to control the interference terms between different waves in the construction. For the Boussinesq system, we do not know if there exist the analogous special solutions. Following an idea in [26, 24], we make use of a multi-steps iteration scheme and one-dimensional oscillation. More precisely, in each step, we add some plane waves which oscillate along the same direction with different frequency, and thus only remove one component of stress error in each step. To reduce the whole stress errors, we divide the process into several steps. On the other hand, since the velocity and temperature are coupled together in Boussinesq system, we need to reduce two stress errors simultaneously. To achieve this, we need to extend the geometric lemma in [7] and add associated plane waves in the velocity and temperature simultaneously in each step. Their coordination is important for us to reduce the temperature stress error and construct continuous temperature.

2. MAIN PROPOSITION AND OUTLINE OF THE PROOF

As in [17], the proof of theorem 1.1 will be achieved through an iteration procedure. In the following, $\mathcal{S}^{2 \times 2}$ always denotes the vector space of symmetric 2×2 matrices.

Definition 2.1. Assume v, p, θ, R, f are smooth and compact supported functions on $R^2 \times R$ taking values, respectively, in $R^2, R, R, \mathcal{S}^{2 \times 2}, R^2$. We say that they solve the Boussinesq-stress system if

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \theta e_2 + \operatorname{div} R, \\ \operatorname{div} v = 0, \\ \theta_t + \operatorname{div}(v \theta) = \operatorname{div} f. \end{cases} \quad (2.1)$$

2.1. Some notations on (semi)norm. In the following, $m = 0, 1, 2, \dots$, β is a multiindex and D is spatial derivative. First of all, we denote the supremum norm $\|f\|_0$ by

$$\|f\|_0 := \sup_{R^2} |f|.$$

Then the \dot{C}^m seminorms are given by

$$[f]_m := \max_{|\beta|=m} \|D^\beta f\|_0$$

and C^m norms are given by

$$\|f\|_m := \sum_{j=0}^m [f]_j.$$

If $f = f_1 + if_2$ is a complex-valued function, then we set $[f]_m := [f_1]_m + [f_2]_m$ and $\|f\|_m := \|f_1\|_m + \|f_2\|_m$.

Moreover, for functions depending on space and time, we introduce the following space-time norm:

$$\|f\|_m := \sup_t \|f(t, \cdot)\|_m, \quad \|f\|_{C_{t,x}^1} := \|f\|_1 + \|\partial_t f\|_0.$$

We now state the main proposition of this paper, of which Theorem 1.1 is a corollary.

Proposition 2.1. *Let $r > 0, \varepsilon > 0$ be two given positive numbers. Then there exist positive constants η, M such that the following property holds:*

For any $0 < \delta \leq 1$, if $(v, p, \theta, R, f) \in C_c^\infty(Q_r)$ solves Boussinesq-stress system (2.1) and

$$\|R\|_0 \leq \eta\delta, \quad (2.2)$$

$$\|f\|_0 \leq \eta\delta. \quad (2.3)$$

Set

$$\Lambda := \max\{1, \|R\|_{C_{t,x}^1}, \|f\|_{C_{t,x}^1}, \|v\|_{C_{t,x}^1}, \|\theta\|_{C_{t,x}^1}\}.$$

Then, for any $\bar{\delta} \leq \frac{1}{2}\delta^{\frac{3}{2}}$, we can construct new functions $(\tilde{v}, \tilde{p}, \tilde{\theta}, \tilde{R}, \tilde{f}) \in C_c^\infty(Q_{r+\delta})$, which also solves Boussinesq-stress system (2.1) and satisfies

$$\|\tilde{R}\|_0 \leq \eta\bar{\delta}, \quad (2.4)$$

$$\|\tilde{f}\|_0 \leq \eta\bar{\delta}, \quad (2.5)$$

$$\|\tilde{v} - v\|_0 \leq M\sqrt{\bar{\delta}}, \quad (2.6)$$

$$\|\tilde{\theta} - \theta\|_0 \leq M\sqrt{\bar{\delta}}, \quad (2.7)$$

$$\|\tilde{p} - p\|_0 \leq M\bar{\delta}, \quad (2.8)$$

and

$$\Lambda_1 := \max\{1, \|\tilde{R}\|_{C_{t,x}^1}, \|\tilde{f}\|_{C_{t,x}^1}, \|\tilde{v}\|_{C_{t,x}^1}, \|\tilde{\theta}\|_{C_{t,x}^1}\} \leq A\delta^{\frac{\varepsilon^2+3\varepsilon+3}{2}} \left(\frac{\sqrt{\delta}}{\bar{\delta}}\right)^{(1+\varepsilon)^2(2+\varepsilon)+(2+\varepsilon)^2} \Lambda^{(1+\varepsilon)^3}. \quad (2.9)$$

Moreover,

$$\|\tilde{p}\|_{C_{t,x}^1} \leq C_0, \quad \|\tilde{\theta}\|_{C_{t,x}^1} \leq A\delta^{\frac{2+\varepsilon}{2}} \left(\frac{\sqrt{\delta}}{\bar{\delta}}\right)^{4+4\varepsilon+\varepsilon^2} \Lambda^{(1+\varepsilon)^2}. \quad (2.10)$$

where the constant A depends on $r, \varepsilon, \|v\|_0$.

We will prove proposition 2.1 in the subsequent sections.

2.2. Outline of the proof of proposition 2.1.

The rest of this paper will be dedicated to prove proposition 2.1. The construction of the functions $\tilde{v}, \tilde{\theta}$ consists of a stage which contains three steps. In the first step, we add perturbations to v_0, θ_0 and get new functions v_{01}, θ_{01} as following:

$$v_{01} = v_0 + w_{1o} + w_{1c} := v_0 + w_1,$$

$$\theta_{01} = \theta_0 + \chi_{1o} + \chi_{1c} := \theta_0 + \chi_1,$$

where $w_{1o}, w_{1c}, \chi_{1o}, \chi_{1c}$ are highly oscillatory functions with compact support given by explicit formulas. We introduce three parameters ℓ, μ_1, λ_1 in the construction of perturbation with

$$1 \ll \mu_1 \ll \lambda_1.$$

After adding these perturbations, we mainly focus on finding functions R_{01}, p_{01} and f_{01} with the desired estimates and solving system (2.1). After the first step, the stresses error become smaller in the following sense:

If

$$R_0(t, x) - e(t, x)Id = -\sum_{i=1}^3 a_i^2(t, x)k_i \otimes k_i, \quad f_0(t, x) = -\sum_{i=1}^2 c_i(t, x)k_i,$$

where $e(t, x)$ is a smooth function with compact support, see (4.6) and (4.7). Then

$$R_{01}(t, x) = -\sum_{i=2}^3 a_i^2(t, x)k_i \otimes k_i + \delta R_{01}, \quad f_{01}(t, x) = -c_2(t, x)k_2 + \delta f_{01},$$

where $\delta R_{01}, \delta f_0$ can be arbitrary small by the appropriate choice of ℓ, μ_1, λ_1 .

We repeat the above step, till obtain the needed $(\tilde{v}, \tilde{p}, \tilde{\theta}, \tilde{R}, \tilde{f})$.

The rest of paper is organized as follows. In section 3, we prove Geometric Lemma and introduce two operators. After these preliminaries, we perform the first step in next three sections. In section 4, we introduce the perturbations $w_{1o}, w_{1c}, \chi_{1o}, \chi_{1c}$ and new stresses R_{01}, f_{01} and prescribe the constant η, M appeared in Proposition 2.1. In sections 5 and 6, we calculate the main forms of R_{01}, f_{01} and prove the relevant estimates of the various terms involved in the construction, in term of the parameters λ_1, μ_1, ℓ . In sections 7, 8 and 9 we construct $v_{0n}, p_{0n}, \theta_{0n}, R_{0n}, f_{0n}$ for $n = 2, 3$ and prove relevant estimates by inductions. The construction given in section 7 is similar to that of the first step in section 4. We calculate the main forms of R_{0n}, f_{0n} in section 8 and prove the various error estimates in section 9. After completing the constructions of $(v_{03}, p_{03}, \theta_{03}, R_{03}, f_{03})$ and various estimates, we give a proof of Proposition 2.1 by choosing appropriate parameters ℓ, μ_n, λ_n for $1 \leq n \leq 3$ in section 10. Finally, in section 11, we give a proof of Theorem 1.1.

3. PRELIMINARIES

We always make use of the following notations: $R^{2 \times 2}$ denotes the space of 2×2 matrices; $\mathcal{S}^{2 \times 2}$, as before, denotes the spaces of 2×2 symmetric matrices and Id denotes 2×2 identity matrix. The matrix norm $|R| := \max_{1 \leq i, j \leq 2} |R_{ij}|$, if $R = (R_{ij})_{2 \times 2}$.

3.1. Geometric Lemma.

The following lemma is a kind of geometric lemma given in [7] to our case. Within it, we represent not only a prescribed symmetric matrix R , but also a prescribed vector.

Lemma 3.1 (Geometric Lemma). *There exist $r_0 > 0$, $k_1, k_2, k_3 \in Z^2 \setminus \{0\}$, smooth positive functions*

$$\gamma_{k_i} \in C^\infty(B_{r_0}(\text{Id})), \quad \frac{1}{2} \leq \gamma_{k_i} \leq \frac{3}{2}, \quad i = 1, 2, 3$$

and linear functions

$$g_{k_i} \in C^\infty(R^2), i = 1, 2$$

such that

(1) for every $R \in B_{r_0}(\text{Id})$, we have

$$R = \sum_{i=1}^3 \gamma_{k_i}^2(R) k_i \otimes k_i;$$

(2) for every $f \in R^2$, we have

$$f = \sum_{i=1}^2 g_{k_i}(f) k_i.$$

Proof. The proof is constructive. We set

$$k_1 := \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)^T, \quad k_2 := \left(-\frac{1}{2}, \frac{2}{\sqrt{6}} \right)^T, \quad k_3 := \left(\frac{1}{2}, 0 \right)^T. \quad (3.1)$$

A straightforward computation gives

$$k_1 \otimes k_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{3} \end{pmatrix}, \quad k_2 \otimes k_2 = \begin{pmatrix} \frac{1}{4} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{3} \end{pmatrix}, \quad k_3 \otimes k_3 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}.$$

It's obvious that $k_1 \otimes k_1, k_2 \otimes k_2, k_3 \otimes k_3$ are linearly independent, hence form a basic for $\mathcal{S}^{2 \times 2}$ and

$$\sum_{i=1}^3 k_i \otimes k_i = Id.$$

Thus, taking $r_0 > 0$ small, for any symmetric matrix $R \in B_{r_0}(Id)$, the following equation

$$\sum_{i=1}^3 \gamma_{k_i}^2 k_i \otimes k_i = R$$

has unique, positive solution γ_{k_i} and

$$\|\gamma_{k_i} - 1\|_0 \leq \frac{1}{2}.$$

Since the representation is unique, the dependence of γ_{k_i} on R is smooth. Then, for any $f \in R^2$, we set

$$\begin{pmatrix} g_{k_1}(f) \\ g_{k_2}(f) \end{pmatrix} := (k_1, k_2)^{-1} f,$$

where $(k_1, k_2)^{-1}$ denote the inverse matrix of (k_1, k_2) . Thus, g_{k_1}, g_{k_2} are linear functions and it's obvious that

$$f = \sum_{i=1}^2 g_{k_i}(f) k_i, \quad \forall f \in R^2.$$

Thus, we finished the proof of this lemma. □

As in [21], we introduce the following operators in order to deal with the stresses.

3.2. The operator \mathcal{R} .

Suppose that the vector function $\vec{U}(x) = (U_1(x), U_2(x))^T \in C_c^\infty(B_r; C^2)$ satisfy

$$\int_{R^2} U_i(x) dx = 0, \quad \int_{R^2} (x_i U_j - x_j U_i)(x) dx = 0, \quad i, j = 1, 2 \quad (3.2)$$

and take the following form

$$\vec{U}(x) = \begin{pmatrix} V_1(x) e^{i\lambda k \cdot x} \\ V_2(x) e^{i\lambda k \cdot x} \end{pmatrix},$$

where $\lambda > 0$, $k = (k_1, k_2) \in R^2 \setminus \{0\}$.

We denote $\mathcal{R}U_o(x)$ by

$$\mathcal{R}U_o(x) := \begin{pmatrix} \frac{M_{11}(x)}{i\lambda} e^{i\lambda k \cdot x} & \frac{M_{12}(x)}{i\lambda} e^{i\lambda k \cdot x} \\ \frac{M_{12}(x)}{i\lambda} e^{i\lambda k \cdot x} & \frac{M_{22}(x)}{i\lambda} e^{i\lambda k \cdot x} \end{pmatrix},$$

where $\vec{M} = (M_{11}, M_{12}, M_{22})$ satisfy

$$M_{11}k_1 + M_{12}k_2 = V_1, \quad M_{12}k_1 + M_{22}k_2 = V_2. \quad (3.3)$$

Obviously, the linear equation (3.3) always have a solution

$$(M_{11}(x), M_{12}(x), M_{22}(x)) \in C_c^\infty(B_r, C^3), \quad \|M_{ij}\|_0 \leq C_0 \|\vec{V}\|_0. \quad (3.4)$$

Here and subsequent in this section, C_0 denotes a absolute constant. In fact, we may choose $M_{12}(x) = 0$ first, if both k_1 and k_2 are not zero, then (3.3) gives (M_{11}, M_{22}) and they satisfy (3.4).

In the case one of k_1, k_2 is zero, for example $k_1 = 0$, (3.3) gives (M_{12}, M_{22}) and we set $M_{11} = 0$. They also satisfy (3.4). We compute

$$\operatorname{div} \mathcal{R}U_o(x) = \begin{pmatrix} V_1(x)e^{i\lambda k \cdot x} \\ V_2(x)e^{i\lambda k \cdot x} \end{pmatrix} + \begin{pmatrix} \frac{\partial_1 M_{11}(x) + \partial_2 M_{12}(x)}{i\lambda} e^{i\lambda k \cdot x} \\ \frac{\partial_1 M_{12}(x) + \partial_2 M_{22}(x)}{i\lambda} e^{i\lambda k \cdot x} \end{pmatrix}.$$

Repeat the above process, there exists $(N_{11}(x), N_{12}(x), N_{22}(x)) \in C_c^\infty(B_r, C^3)$ such that if we set

$$\mathcal{R}U_{c1}(x) := \begin{pmatrix} \frac{N_{11}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} & \frac{N_{12}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} \\ \frac{N_{12}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} & \frac{N_{22}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} \end{pmatrix},$$

then

$$\operatorname{div} \mathcal{R}U_{c1}(x) = - \begin{pmatrix} \frac{\partial_1 M_{11}(x) + \partial_2 M_{12}(x)}{i\lambda} e^{i\lambda k \cdot x} \\ \frac{\partial_1 M_{12}(x) + \partial_2 M_{22}(x)}{i\lambda} e^{i\lambda k \cdot x} \end{pmatrix} + \begin{pmatrix} \frac{\partial_1 N_{11}(x) + \partial_2 N_{12}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} \\ \frac{\partial_1 N_{12}(x) + \partial_2 N_{22}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} \end{pmatrix}$$

and

$$\|N_{ij}\|_0 \leq C_0 \|\nabla M\|_0 \leq C_0 \|\nabla V\|_0, \quad \|N_{ij}\|_1 \leq C_0 \|\nabla^2 V\|_0.$$

A straightforward computation gives

$$\operatorname{div}(\mathcal{R}U_o(x) + \mathcal{R}U_{c1}(x)) = \begin{pmatrix} V_1(x)e^{i\lambda k \cdot x} \\ V_2(x)e^{i\lambda k \cdot x} \end{pmatrix} + \begin{pmatrix} \frac{\partial_1 N_{11}(x) + \partial_2 N_{12}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} \\ \frac{\partial_1 N_{12}(x) + \partial_2 N_{22}(x)}{(i\lambda)^2} e^{i\lambda k \cdot x} \end{pmatrix}.$$

Performing the above process, for any integer $m \geq 2$, we have symmetric matrix functions $\mathcal{R}U_{ci} \in C_c^\infty(B_r) : i = 1, 2, \dots, m-1$ such that

$$\|\mathcal{R}U_{ci}\|_0 \leq C_0 \frac{\|\nabla^i V\|_0}{\lambda^{i+1}}$$

and

$$\operatorname{div}\left(\mathcal{R}U_o(x) + \sum_{i=1}^{m-1} \mathcal{R}U_{ci}(x)\right) = \begin{pmatrix} V_1(x)e^{i\lambda k \cdot x} \\ V_2(x)e^{i\lambda k \cdot x} \end{pmatrix} + \begin{pmatrix} \frac{R_1}{(i\lambda)^m} e^{i\lambda k \cdot x} \\ \frac{R_2}{(i\lambda)^m} e^{i\lambda k \cdot x} \end{pmatrix}$$

with

$$\|R_1\|_0 + \|R_2\|_0 \leq C_0 \|\nabla^m V\|_0.$$

Since $\mathcal{R}U_o(x) + \sum_{i=1}^{m-1} \mathcal{R}U_{ci}(x) \in C_c^\infty(B_r)$ is a symmetric matrix, then

$$\int_{R^2} \operatorname{div}\left(\mathcal{R}U_o(x) + \sum_{i=1}^{m-1} \mathcal{R}U_{ci}(x)\right) = 0, \quad \int_{R^2} (x_i H_j - x_j H_i)(x) dx = 0, \quad i, j = 1, 2. \quad (3.5)$$

Here we used the notation $\operatorname{div}\left(\mathcal{R}U_o(x) + \sum_{i=1}^{m-1} \mathcal{R}U_{ci}(x)\right) = (H_1, H_2)^T$.

By (3.2) and (3.5), if we set

$$(K_1, K_2)^T := \left(\frac{R_1}{(i\lambda)^m} e^{i\lambda k \cdot x}, \frac{R_2}{(i\lambda)^m} e^{i\lambda k \cdot x} \right)^T,$$

we also have

$$K_i \in C_c^\infty(B_r; C), \quad \int_{R^2} K_i(x) dx = 0, \quad \int_{R^2} (x_i K_j - x_j K_i)(x) dx = 0, \quad i, j = 1, 2.$$

Together with a result given in [21], we know that there exists a symmetric matrix function $\delta\mathcal{R}U \in C_c^\infty(B_r)$ such that

$$\operatorname{div}\delta\mathcal{R}U = -(K_1, K_2)^T, \quad \|\delta\mathcal{R}U\|_0 \leq C_0(r)\|K\|_0 \leq C_0(r)\frac{\|\nabla^m V\|_0}{\lambda^m}.$$

Finally, we set $\mathcal{R}U := \mathcal{R}U_0 + \sum_{i=1}^{m-1} \mathcal{R}U_{ci}(x) + \delta\mathcal{R}U$, then $\mathcal{R}U$ is a symmetric matrix function and satisfies

$$\mathcal{R}U \in C_c^\infty(B_r), \quad \operatorname{div}\mathcal{R}U = \vec{U}.$$

Moreover, there holds $\|\mathcal{R}U\|_0 \leq C_0(r)\left(\sum_{i=0}^{m-1} \frac{\|V\|_i}{\lambda^{i+1}} + \frac{\|V\|_m}{\lambda^m}\right)$. In fact,

$$\|\mathcal{R}U_0\|_0 \leq C_0 \frac{\|V\|_0}{\lambda}, \quad \|\mathcal{R}U_{ci}\|_0 \leq C_0 \frac{\|V\|_i}{\lambda^{i+1}}, i = 1, \dots, m-1, \quad \|\delta\mathcal{R}U\|_0 \leq C_0(r) \frac{\|V\|_m}{\lambda^m}.$$

Sum them is what we claimed.

Now we introduce a vector space. Put

$$\Xi := \left\{ \vec{U}(x) : \vec{U}(x) = (U_1(x), U_2(x))^T \in C_c^\infty(B_r; C^2), \int_{R^2} U_i(x) dx = 0, \right. \\ \left. \int_{R^2} (x_i U_j - x_j U_i)(x) dx = 0, \quad i, j = 1, 2 \quad \text{and} \quad \vec{U}(x) = \sum_{j=0}^7 \begin{pmatrix} U_{1j}(x) e^{i\lambda_j k \cdot x} \\ U_{2j}(x) e^{i\lambda_j k \cdot x} \end{pmatrix} \right\},$$

where $k \in R^2 \setminus \{0\}$ and $\lambda_j > 0, j = 0, \dots, 7$.

Proposition 3.1. *There exists a linear operator \mathcal{R} from Ξ to $C_c^\infty(B_r; \mathcal{S}^{2 \times 2})$ such that for any $\vec{U}(x) \in \Xi$ with*

$$\vec{U}(x) = \sum_{j=0}^7 \vec{U}_j(x) := \sum_{j=0}^7 \begin{pmatrix} U_{1j}(x) e^{i\lambda_j k \cdot x} \\ U_{2j}(x) e^{i\lambda_j k \cdot x} \end{pmatrix},$$

there holds

$$\operatorname{div}\mathcal{R}U(x) = \vec{U}(x), \quad \|\mathcal{R}U\|_0 \leq C_0(r) \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\|U_{1j}\|_i + \|U_{2j}\|_i}{\lambda_j^{i+1}} + \frac{\|U_{1j}\|_m + \|U_{2j}\|_m}{\lambda_j^m} \right). \quad (3.6)$$

Proof. We have defined the operator \mathcal{R} on function $\vec{U}_j(x) = \begin{pmatrix} U_{1j}(x) e^{i\lambda_j k \cdot x} \\ U_{2j}(x) e^{i\lambda_j k \cdot x} \end{pmatrix}$ and

$$\|\mathcal{R}U_j\|_0 \leq C_0(r) \left(\sum_{i=0}^{m-1} \frac{\|U_{1j}\|_i + \|U_{2j}\|_i}{\lambda_j^{i+1}} + \frac{\|U_{1j}\|_m + \|U_{2j}\|_m}{\lambda_j^m} \right).$$

Then, set

$$\mathcal{R}U := \sum_{j=0}^7 \mathcal{R}U_j.$$

It is obvious that $\mathcal{R}U \in C_c^\infty(B_r; \mathcal{S}^{2 \times 2})$ and satisfies (3.6). □

3.3. The operator \mathcal{G} .

Let $f(x) = \varphi(x)e^{i\lambda k \cdot x}$ with $\varphi(x) \in C_c^\infty(B_r; \mathbb{C})$ and $\int_{\mathbb{R}^2} \varphi(x)e^{i\lambda k \cdot x} dx = 0$, where $\lambda > 0$ and $k \in \mathbb{R}^2 \setminus \{0\}$. Set

$$\mathcal{G}f_o := \frac{k}{i\lambda|k|^2} \varphi(x)e^{i\lambda k \cdot x},$$

then

$$\operatorname{div} \mathcal{G}f_o = f + \frac{k \cdot \nabla \varphi}{i\lambda|k|^2} e^{i\lambda k \cdot x}.$$

Set

$$\mathcal{G}f_{c1} := -\frac{k}{(i\lambda)^2|k|^4} k \cdot \nabla \varphi e^{i\lambda k \cdot x},$$

then

$$\operatorname{div}(\mathcal{G}f_o + \mathcal{G}f_{c1}) = f - \frac{(k \cdot \nabla)^2 \varphi}{(i\lambda)^2|k|^4} e^{i\lambda k \cdot x}.$$

It's obvious that

$$\|\mathcal{G}f_o\|_0 \leq C_0 \frac{\|\varphi\|_0}{\lambda}, \quad \|\mathcal{G}f_{c1}\|_0 \leq C_0 \frac{\|\varphi\|_1}{\lambda^2}.$$

Performing this process, for any integer $m \geq 2$, there exist $\mathcal{G}f_{ci} : i = 1, 2, \dots, m-1$ such that

$$\operatorname{div}(\mathcal{G}f_o + \sum_{i=1}^{m-1} \mathcal{G}f_{ci}) = f + (-1)^{m-1} \frac{(k \cdot \nabla)^m \varphi}{(i\lambda)^m|k|^{2m}} e^{i\lambda k \cdot x}$$

and

$$\|\mathcal{G}f_{ci}\|_0 \leq C_0 \frac{\|\varphi\|_i}{\lambda^{i+1}}.$$

Since

$$\int_{\mathbb{R}^2} f(x) dx = 0, \quad \int_{\mathbb{R}^2} \operatorname{div} \left(\mathcal{G}f_o + \sum_{i=0}^{m-1} \mathcal{G}f_{ci} \right) dx = 0,$$

therefore

$$\int_{\mathbb{R}^2} \frac{(k \cdot \nabla)^m \varphi}{(i\lambda|k|^2)^m} e^{i\lambda k \cdot x} dx = 0.$$

By a result in [29], see Chapter 1, we know that there exists $\mathcal{G}f_{cm} \in C_c^\infty(B_r; \mathbb{C}^2)$ such that

$$\operatorname{div} \mathcal{G}f_{cm} = (-1)^m \frac{(k \cdot \nabla)^m \varphi}{(i\lambda)^m|k|^{2m}} e^{i\lambda k \cdot x}, \quad \|\mathcal{G}f_{cm}\|_0 \leq C_0(r) \frac{\|\nabla^m \varphi\|_0}{\lambda^m}.$$

In fact, there exists $\mathcal{G}f_{cm}$ such that

$$\|\nabla \mathcal{G}f_{cm}\|_4 \leq C_0(r) \left\| \frac{(k \cdot \nabla)^m \varphi}{(i\lambda)^m|k|^{2m}} \right\|_4 \leq C_0(r) \frac{\|\nabla^m \varphi\|_0}{\lambda^m}.$$

Moreover, due to $\mathcal{G}f_{cm} \in C_c^\infty(B_r)$, we have $\|\mathcal{G}f_{cm}\|_{W^{1,4}} \leq C_0(r) \frac{\|\nabla^m \varphi\|_0}{\lambda^m}$. Sobolev embedding gives $\|\mathcal{G}f_{cm}\|_0 \leq C_0(r) \frac{\|\nabla^m \varphi\|_0}{\lambda^m}$ which is what we claimed.

Finally, we set

$$\mathcal{G}f := \mathcal{G}f_o + \sum_{i=1}^m \mathcal{G}f_{ci},$$

then

$$\operatorname{div} \mathcal{G}f = f$$

and

$$\mathcal{G}f \in C_c^\infty(B_r; C^2), \quad \|\mathcal{G}f\|_0 \leq C_0(r) \left(\sum_{i=0}^{m-1} \frac{\|\varphi\|_i}{\lambda^{i+1}} + \frac{\|\varphi\|_m}{\lambda^m} \right).$$

In conclusion, we have

Proposition 3.2. *Let the vector space Ψ given by*

$$\Psi := \left\{ H(x) : H(x) = \sum_{j=0}^m H_j(x) := \sum_{j=0}^m b_j(x) e^{i\lambda_j k \cdot x}, \quad b_j \in C_c^\infty(B_r; C) \quad \text{and} \quad \int_{R^2} H_j(x) dx = 0 \right\},$$

then there exists a linear operator $\mathcal{G} : \Psi \rightarrow C_c^\infty(B_r; C^2)$ such that for any positive integer m and any $H(x) = \sum_{j=0}^7 b_j(x) e^{i\lambda_j k \cdot x} \in \Psi$, there holds

$$\operatorname{div} \mathcal{G}(H)(x) = H(x), \quad \|\mathcal{G}(H)\|_0 \leq C_0(r) \sum_{j=0}^7 \sum_{i=0}^{m-1} \left(\frac{\|b_j\|_i}{\lambda_j^{i+1}} + \frac{\|b_j\|_m}{\lambda_j^m} \right). \quad (3.7)$$

Proof. The proof is similar to that of Proposition 3.1, we omit it here. \square

4. THE CONSTRUCTION OF APPROXIMATE SOLUTIONS

The construction of \tilde{v} , \tilde{p} , $\tilde{\theta}$, \tilde{R} , \tilde{g} from v , p , θ , R , g consists of several steps. The main idea is to decompose the stress errors into some blocks with the add of geometric lemma and remove one block by constructing new approximate solutions in each step. In this section, we perform the first step.

For convenience, we set $v_0 := v$, $p_0 := p$, $\theta_0 := \theta$, $R_0 := R$, $g_0 := g$ and in this section C_0 denotes a absolute constant.

4.1. Partition of unity and Conditions on the parameters. We first introduce a partition of unity. Following the construction given in [17], we have the following partition of unity. For two constants c_1 and c_2 such that $\frac{\sqrt{3}}{2} < c_1 < c_2 < 1$, we have a family of functions $\alpha_l \in C_c^\infty(R^3)$, $l \in Z^3$ such that

$$\sum_{l \in Z^3} \alpha_l^2 = 1, \quad \operatorname{supp} \alpha_l \subseteq B_{c_2}(l). \quad (4.1)$$

Our construction depends on three parameters: ℓ, μ_1, λ_1 and we assume they satisfy the following inequalities:

$$\mu_1 \geq \frac{\Lambda}{\delta} \geq 1, \quad \ell^{-1} \geq \frac{\Lambda}{\eta \delta} \geq 1, \quad \lambda_1 \geq \max\{\mu_1^{1+\varepsilon}, \ell^{-(1+\varepsilon)}\}. \quad (4.2)$$

4.2. Decomposition of stress error. First, we apply Geometric Lemma 3.1 to obtain $r_0 > 0$ and vectors

$$k_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)^T, \quad k_2 = \left(-\frac{1}{2}, \frac{2}{\sqrt{6}} \right)^T, \quad k_3 = \left(\frac{1}{2}, 0 \right)^T$$

which are given in the proof of lemma 3.1, see (3.1), together with corresponding functions

$$\gamma_{k_i} \in C^\infty(B_{r_0}(Id)), \quad g_{k_i} \in C^\infty(R^2), \quad i = 1, 2, 3,$$

where $g_{k_3} = 0$. Next, we let $\varphi \in C_c^\infty(R^2 \times R)$ be a standard nonnegative radial function and denote the corresponding family of mollifiers by

$$\varphi_\ell(t, x) := \frac{1}{\ell^3} \varphi\left(\frac{t}{\ell}, \frac{x}{\ell}\right).$$

Then set

$$f_{0\ell}(t, x) := f_0 * \varphi_\ell(t, x), \quad R_{0\ell}(t, x) := R_0 * \varphi_\ell(t, x).$$

By Lemma 3.1, we decompose f_0 as

$$f_0(t, x) = \sum_{i=1}^2 g_{k_i}(f_0)(t, x) k_i := - \sum_{i=1}^2 c_i(t, x) k_i.$$

Here we denote $c_i(t, x) := -g_{k_i}(f_0)(t, x)$. Thus

$$f_{0\ell}(t, x) = - \sum_{i=1}^2 c_{i\ell}(t, x) k_i. \quad (4.3)$$

Since g_{k_1} is linear function,

$$c_{i\ell}(t, x) = -g_{k_i}(f_{0\ell})(t, x). \quad (4.4)$$

By (2.3), we know that

$$c_{i\ell}(t, x) \in C_c^\infty(Q_{r+\ell}), \quad \|c_i\|_0 \leq 2\delta. \quad (4.5)$$

Then, we introduce $\rho(t, x)$ as following

$$\rho(t, x) \in C_c^\infty(Q_{r+\delta}), \quad \rho(t, x) = \sqrt{2\delta} \text{ in } Q_{r+\frac{\delta}{2}}, \quad 0 \leq \rho(t, x) \leq \sqrt{2\delta}, \quad \|\rho\|_{C_{t,x}^k} \leq C_0 \delta^{-\frac{1}{2}-(k-1)} \quad (4.6)$$

and set

$$e(t, x) := \rho^2(t, x). \quad (4.7)$$

By (2.2), parameter assumption (4.2), (4.6) and (4.7), we have

$$\left\| \frac{R_{0\ell}(t, \cdot)}{e(t, \cdot)} \right\|_0 \leq \frac{\eta}{2} \leq \frac{r_0}{2}$$

if we take $\eta = r_0$, thus by Lemma 3.1

$$\begin{aligned} e(t, x) Id - R_{0\ell}(t, x) &= e(t, x) \left(Id - \frac{R_{0\ell}(t, x)}{e(t, x)} \right) = e(t, x) \sum_{i=1}^3 \gamma_{k_i}^2 \left(Id - \frac{R_{0\ell}(t, x)}{e(t, x)} \right) k_i \otimes k_i \\ &= \sum_{i=1}^3 \left(\rho(t, x) \gamma_{k_i} \left(Id - \frac{R_{0\ell}(t, x)}{e(t, x)} \right) \right)^2 k_i \otimes k_i := \sum_{i=1}^3 a_i^2(t, x) k_i \otimes k_i. \end{aligned} \quad (4.8)$$

Where $a_i(t, x) = \rho(t, x) \gamma_{k_i} \left(Id - \frac{R_{0\ell}(t, x)}{e(t, x)} \right)$ satisfies

$$a_i \in C_c^\infty(Q_{r+\delta}), \quad \|a_i\|_0 \leq \frac{M\sqrt{\delta}}{300}. \quad (4.9)$$

Here we denote constant M by

$$M := \max \left\{ C_0, 600 \max_{1 \leq i \leq 3} \|\gamma_{k_i}\|_{L^\infty(B_{\frac{r_0}{2}}(Id))}, 600 \max_{1 \leq i \leq 3} \frac{1}{\min_{B_{\frac{r_0}{2}}(Id)} \gamma_{k_i}} \right\}. \quad (4.10)$$

4.3. Construction of 1-th perturbation on velocity.

4.3.1. *Main perturbation on velocity.* For any $l \in Z^3$, we set

$$b_{1l}(t, x) := \frac{a_1(t, x)\alpha_l(\mu_1 t, \mu_1 x)}{\sqrt{2}}, \quad (4.11)$$

then, by (4.9), it's easy to obtain

$$\|b_{1l}\|_0 \leq \frac{M\sqrt{\delta}}{300}. \quad (4.12)$$

As in [24], we set $[l] := \sum_{j=0}^2 2^j [l_j]$, if $l = (l_0, l_1, l_2)$, where

$$[l_j] = \begin{cases} 1, & l_j \text{ is even,} \\ 0, & l_j \text{ is odd.} \end{cases}$$

Thus, $[l]$ can only take values in $\{0, 1, \dots, 7\}$.

Now we denote main l -perturbation w_{1ol} by

$$w_{1ol}(t, x) := b_{1l}(t, x)k_1 \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} + e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} \right). \quad (4.13)$$

Here and subsequent, we denote $a^\perp = (-a_2, a_1)^T$ if $a = (a_1, a_2)^T$.

Then set 1-th main perturbation

$$w_{1o} := \sum_{l \in Z^3} w_{1ol} = \sum_{j=0}^7 \sum_{[l]=j} b_{1l} k_1 \left(e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} + e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} \right). \quad (4.14)$$

Obviously, w_{1ol}, w_{1o} are all real 2-dimensional vector-valued functions.

By (4.1), we have $\text{supp}\alpha_l \cap \text{supp}\alpha_{l'} = \emptyset$ if $|l - l'| \geq 2$, hence there are at most 30 nonzero terms at every point $(t, x) \in R^3$ in the summation (4.14), thus by (4.12)

$$\|w_{1o}\|_0 \leq \frac{M\sqrt{\delta}}{6}. \quad (4.15)$$

Furthermore, if $b_{1l}(t, x) \neq 0$, then $|(\mu_1 t, \mu_1 x) - l| \leq 1$ and $|(t, x)| \leq r + \delta$, thus $|l| \leq C_0(r)\mu_1$. By (4.9), we know that for any $l \in Z^3$,

$$b_{1l} \in C_c^\infty(Q_{r+\delta}), \quad w_{1ol} \in C_c^\infty(Q_{r+\delta}), \quad w_{1o} \in C_c^\infty(Q_{r+\delta}). \quad (4.16)$$

4.3.2. *The correction w_{1c} and the 1-th perturbation w_1 .* We define l -correction by

$$\begin{aligned} w_{1cl} := & \frac{\nabla^\perp b_{1l}}{i\lambda_1 2^{[l]}} \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} - e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} \right) \\ & - \nabla^\perp \left(\frac{\nabla b_{1l} \cdot k_1^\perp}{\lambda_1^2 2^{2[l]} |k_1|^2} \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} + e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{t}{\mu_1})t)} \right) \right), \end{aligned} \quad (4.17)$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})^T$.

Then 1-th correction is given by

$$w_{1c} := \sum_{l \in Z^3} w_{1cl}. \quad (4.18)$$

Finally, we denote 1-th perturbation w_1 by

$$w_1 := w_{1o} + w_{1c}.$$

Thus, if we denote w_{1l} by

$$w_{1l} := w_{1ol} + w_{1cl},$$

then, we have

$$w_{1l} = \nabla^\perp \operatorname{div} \left(-\frac{b_{1l}}{\lambda_1^2 2^{2[l]} |k_1|^2} k_1^\perp 2 \cos \left(\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right) \right) \right) \quad (4.19)$$

and

$$w_1 = \sum_{l \in \mathbb{Z}^3} w_{1l}, \quad \operatorname{div} w_{1l} = 0, \quad \int_{R^2} w_{1l} dx = 0, \quad \int_{R^2} (x_i w_{1lj} - x_j w_{1li}) dx = 0, \quad i, j = 1, 2.$$

In fact

$$\int_{R^2} (x_1 w_{1l2} - x_2 w_{1l1}) dx = \int_{R^2} (x_1 \partial_1 \operatorname{div} \vec{a}_1 + x_2 \partial_2 \operatorname{div} \vec{a}_1) dx = -2 \int_{R^2} \operatorname{div} \vec{a}_1 dx = 0,$$

where

$$\vec{a}_1 = -\frac{b_{1l}}{\lambda_1^2 2^{2[l]} |k_1|^2} k_1^\perp 2 \cos \left(\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right) \right)$$

is a vector-valued smooth function with compact support.

Since there are only finite nonzero terms in the summation (4.14) and (4.18), thus

$$\operatorname{div} w_1 = 0, \quad \int_{R^2} w_1 dx = 0, \quad \int_{R^2} (x_i w_{1j} - x_j w_{1i}) dx = 0, \quad i, j = 1, 2. \quad (4.20)$$

Now we set

$$\begin{aligned} k_{1l} &:= b_{1l} k_1 + \frac{\nabla^\perp b_{1l}}{i \lambda_1 2^{[l]}} - \nabla^\perp \left(\frac{\nabla b_{1l} \cdot k_1^\perp}{\lambda_1^2 2^{2[l]} |k_1|^2} \right) + \frac{\nabla b_{1l} \cdot k_1^\perp}{i \lambda_1 2^{[l]} |k_1|^2} \cdot k_1, \\ k_{-1l} &:= b_{1l} k_1 + \frac{\nabla^\perp b_{1l}}{-i \lambda_1 2^{[l]}} - \nabla^\perp \left(\frac{\nabla b_{1l} \cdot k_1^\perp}{\lambda_1^2 2^{2[l]} |k_1|^2} \right) + \frac{\nabla b_{1l} \cdot k_1^\perp}{-i \lambda_1 2^{[l]} |k_1|^2} \cdot k_1, \end{aligned} \quad (4.21)$$

then

$$w_{1l} = k_{1l} e^{i \lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} + k_{-1l} e^{-i \lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)}$$

and

$$w_1 = \sum_{j=0}^7 \sum_{[l]=j} \left(k_{1l} e^{i \lambda_1 2^j k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} + k_{-1l} e^{-i \lambda_1 2^j k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right). \quad (4.22)$$

Finally, it's obvious

$$w_{1ol}, w_{1cl}, w_{1l}, w_{1o}, w_{1c}, w_1, k_{1l}, k_{-1l} \in C_c^\infty(Q_{r+\delta}).$$

Thus, we complete the construction of perturbation w_1 .

4.4. Construction of 1-th perturbation on temperature.

To construct χ_1 , we first denote β_{1l} by

$$\beta_{1l}(t, x) := \frac{c_{1l}(t, x) \alpha_1(\mu_1 t, \mu_1 x)}{\sqrt{2e(t, x)} \gamma_{k_1} \left(Id - \frac{R_{0l}(t, x)}{e(t, x)} \right)}. \quad (4.23)$$

Since $\operatorname{supp} c_{1l} \subseteq \operatorname{supp} e$, so β_{1l} is well-defined. Then we denote main l -perturbation χ_{1ol} by

$$\chi_{1ol}(t, x) := \beta_{1l}(t, x) \left(e^{i \lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} + e^{-i \lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right)$$

and l -correction χ_{1cl} by

$$\begin{aligned} \chi_{1cl}(t, x) := & \triangle \beta_{1l}(t, x) \left(\frac{e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}}{-\lambda_1^2 2^{2[l]} |k_1|^2} + \frac{e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}}{-\lambda_1^2 2^{2[l]} |k_1|^2} \right) \\ & + 2\nabla \beta_{1l}(t, x) \cdot \nabla \left(\frac{e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}}{-\lambda_1^2 2^{2[l]} |k_1|^2} + \frac{e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}}{-\lambda_1^2 2^{2[l]} |k_1|^2} \right). \end{aligned}$$

Finally, the l -th perturbation is given by

$$\chi_{1l}(t, x) := \chi_{1ol}(t, x) + \chi_{1cl}(t, x) = \triangle \left(\beta_{1l}(x, t) \left(\frac{e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}}{-\lambda_1^2 2^{2[l]} |k_1|^2} + \frac{e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}}{-\lambda_1^2 2^{2[l]} |k_1|^2} \right) \right).$$

Set

$$\chi_{1o}(t, x) := \sum_{l \in Z^3} \chi_{1ol}(t, x), \quad \chi_{1c}(t, x) := \sum_{l \in Z^3} \chi_{1cl}(t, x), \quad \chi_1(t, x) := \sum_{l \in Z^3} \chi_{1l}(t, x).$$

Obviously, χ_{1ol} , χ_{1cl} , χ_{1l} and χ_1 are all real scalar functions and

$$\chi_{1o}(t, x) = \sum_{j=0}^7 \sum_{[l]=j} \beta_{1l}(t, x) \left(e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right).$$

Moreover, it's easy to get

$$\int_{R^2} \chi_{1l}(t, x) dx = 0, \quad \int_{R^2} x_1 \chi_{1l}(t, x) dx = 0.$$

Since there are only finite terms in the summation of χ_1 , therefore

$$\int_{R^2} \chi_1(t, x) dx = 0, \quad \int_{R^2} x_1 \chi_1(t, x) dx = 0. \quad (4.24)$$

If set

$$h_{1l} := \beta_{1l} - \frac{\triangle \beta_{1l}}{\lambda_1^2 2^{2[l]} |k_1|^2} + 2 \frac{\nabla \beta_{1l} \cdot k_1^\perp}{i\lambda_1 2^{[l]} |k_1|^2}, \quad h_{-1l} := \beta_{1l} - \frac{\triangle \beta_{1l}}{\lambda_1^2 2^{2[l]} |k_1|^2} + 2 \frac{\nabla \beta_{1l} \cdot k_1^\perp}{-i\lambda_1 2^{[l]} |k_1|^2}, \quad (4.25)$$

then

$$\chi_{1l} = h_{1l} e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + h_{-1l} e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}$$

and

$$\chi_1 = \sum_{j=0}^7 \sum_{[l]=j} \left(h_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + h_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right). \quad (4.26)$$

Since $c_{1l}(t, x) \in C_c^\infty(Q_{r+\delta})$, we know that for all $l \in Z^3$

$$\beta_{1l} \in C_c^\infty(Q_{r+\delta}), \quad h_{1l} \in C_c^\infty(Q_{r+\delta})$$

and

$$\chi_{1ol}, \quad \chi_{1cl}, \quad \chi_{1l}, \quad \chi_1 \in C_c^\infty(Q_{r+\delta}).$$

Then, by (4.5), (4.6), (4.7), (4.10) and (4.23), we know that

$$\|\beta_{1l}\|_0 \leq \frac{M\sqrt{\delta}}{200}.$$

Similar to (4.15), we have

$$\|\chi_{1o}\|_0 \leq \frac{M\sqrt{\delta}}{4}. \quad (4.27)$$

4.5. The construction of v_{01} , p_{01} , θ_{01} , R_{01} , f_{01} .

First, we denote M_1 by

$$M_1 := \sum_{l \in Z^3} b_{1l}^2 k_1 \otimes k_1 \left(e^{2i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-2i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) + \sum_{l, l' \in Z^3, l \neq l'} w_{1ol} \otimes w_{1ol'}$$

and N_1, K_1 by

$$\begin{aligned} N_1 &:= \sum_{l \in Z^3} \left[w_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) + \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \otimes w_{1l} \right] + R_0 - R_{0\ell}, \\ K_1 &:= \sum_{l \in Z^3} \beta_{1l} b_{1l} k_1 \left(e^{2i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-2i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) + \sum_{l, l' \in Z^3, l \neq l'} w_{1ol} \chi_{1ol'}. \end{aligned}$$

Then set

$$\begin{aligned} v_{01}(t, x) &:= v_0(t, x) + w_1(t, x), \quad p_{01}(t, x) := p_0(t, x) - e(t, x), \quad \theta_{01}(t, x) := \theta_0(t, x) + \chi_1(t, x), \\ R_{01}(t, x) &:= -\bar{R}_{0\ell}(t, x) + 2 \sum_{l \in Z^3} b_{1l}^2(t, x) k_1 \otimes k_1 + \delta R_{01}(t, x), \\ f_{01}(t, x) &:= f_{0\ell}(t, x) + 2 \sum_{l \in Z^3} \beta_{1l}(t, x) b_{1l}(t, x) k_1 + \delta f_{01}(t, x), \end{aligned} \quad (4.28)$$

where

$$\bar{R}_{0\ell}(t, x) = -R_{0\ell}(t, x) + e(t, x)Id,$$

$$\begin{aligned} \delta R_{01} &:= \mathcal{R}(\operatorname{div} M_1) + N_1 - \mathcal{R}(\chi_1 e_2) + \mathcal{R} \left\{ \partial_t w_1 + \operatorname{div} \left[\sum_{l \in Z^3} \left(w_{1l} \otimes v_0 \left(\frac{l}{\mu_1} \right) + v_0 \left(\frac{l}{\mu_1} \right) \otimes w_{1l} \right) \right] \right\} \\ &\quad + (w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}) \end{aligned} \quad (4.29)$$

and

$$\begin{aligned} \delta f_{01} &:= \mathcal{G}(\operatorname{div} K_1) + \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) + w_{1o} \chi_{1c} + f_0 - f_{0\ell} + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} \\ &\quad + w_{1c} \chi_1 + \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_2} \right) \right). \end{aligned} \quad (4.30)$$

By (4.20) and (4.24), we know

$$\operatorname{div} M_1, \chi_1 e_2, \partial_t w_1, \operatorname{div} \left[\sum_{l \in Z^3} \left(w_{1l} \otimes v_0 \left(\frac{l}{\mu_1} \right) + v_0 \left(\frac{l}{\mu_1} \right) \otimes w_{1l} \right) \right] \in \Xi,$$

so δR_{01} is well-defined. Notice that

$$\operatorname{div} K_1, \partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \in \Psi,$$

thus δf_{01} is also well-defined. By Proposition 3.1, we know that δR_{01} is a symmetric matrix and $\delta R_{01} \in C_c^\infty(Q_{r+\delta})$. Also, by Proposition 3.2, we have $\delta f_{01} \in C_c^\infty(Q_{r+\delta})$.

Obviously,

$$\operatorname{div} v_{01} = \operatorname{div} v_0 + \operatorname{div} w_1 = 0.$$

Moreover, from the definition of $(v_{01}, p_{01}, \theta_{01}, R_{01}, f_{01})$ and the fact that $(v_0, p_0, \theta_0, R_0, f_0)$ solves the system (2.1), together with Proposition 3.1 we know that

$$\begin{aligned} \operatorname{div} R_{01} &= \operatorname{div} R_0 - \nabla e + \partial_t w_1 - \chi_1 e_2 + \operatorname{div}(w_{1o} \otimes w_{1o} + w_1 \otimes v_0 + v_0 \otimes w_1 \\ &\quad + w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}) \\ &= \partial_t v_0 + \operatorname{div}(v_0 \otimes v_0) + \nabla p_0 - \theta_0 e_2 - \nabla e + \partial_t w_1 - \chi_1 e_2 + \operatorname{div}(w_{1o} \otimes w_{1o} \\ &\quad + w_1 \otimes v_0 + v_0 \otimes w_1 + w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}) \\ &= \partial_t v_{01} + \operatorname{div}(v_{01} \otimes v_{01}) + \nabla p_{01} - \theta_{01} e_2. \end{aligned}$$

Here we use the fact

$$\operatorname{div}(M_1) + \operatorname{div}\left(2 \sum_{l \in Z^3} b_{1l}^2(x, t) k_1 \otimes k_1\right) = \operatorname{div}(w_{1o} \otimes w_{1o}).$$

Furthermore, by (4.28) and (4.30), we have

$$\begin{aligned} f_{01} &= f_0 + 2 \sum_{l \in Z^3} \beta_{1l} b_{1l} k_1 + \mathcal{G}(\operatorname{div} K_1) + w_{1o} \chi_{1c} + \mathcal{G}\left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1}\right) \cdot \nabla \chi_{1l}\right) \\ &\quad + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1}\right)\right) \chi_{1l} + w_{1c} \chi_1 + \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_2}\right)\right). \end{aligned}$$

Thus, by Proposition 3.2 and the fact that $(v_0, p_0, \theta_0, R_0, f_0)$ solves the system (2.1), we have

$$\begin{aligned} \operatorname{div} f_{01} &= \operatorname{div} f_0 + \partial_t \chi_1 + \operatorname{div}(w_{1o} \chi_1 + w_{1c} \chi_1 + v_0 \chi_1 + w_1 \theta_0) \\ &= \operatorname{div}(v_0 \theta_0 + w_{1o} \chi_1 + w_{1c} \chi_1 + v_0 \chi_1 + w_1 \theta_0) + \partial_t(\theta_0 + \chi_1) \\ &= \partial_t \theta_{01} + \operatorname{div}(v_{01} \theta_{01}). \end{aligned}$$

Here we use the fact

$$\operatorname{div} K_1 + \operatorname{div}\left(2 \sum_{l \in Z^3} \beta_{1l}(x, t) b_{1l}(x, t) k_1\right) = \operatorname{div}(w_{1o} \chi_{1o}).$$

Thus the new functions $(v_{01}, p_{01}, \theta_{01}, R_{01}, f_{01})$ solves the system (2.1).

5. THE 1-TH REPRESENTATIONS

In this section, we will calculate the following two terms

$$-\bar{R}_{0\ell} + 2 \sum_{l \in Z^3} b_{1l}^2 k_1 \otimes k_1 = I$$

and

$$f_{0\ell} + 2 \sum_{l \in Z^3} \beta_{1l} b_{1l} k_1 = II.$$

5.1. The term I.

First, by (4.11) and the fact $\sum_{l \in Z^3} \alpha_l^2 = 1$, we have

$$2 \sum_{l \in Z^3} b_{1l}^2(t, x) k_1 \otimes k_1 = \sum_{l \in Z^3} \alpha_l^2(\mu_1 t, \mu_1 x) a_1^2(t, x) k_1 \otimes k_1 = a_1^2(t, x) k_1 \otimes k_1.$$

Thus, by (4.8),

$$-\bar{R}_{0\ell}(x, t) + 2 \sum_{l \in Z^3} b_{1l}^2(x, t) k_1 \otimes k_1 = - \sum_{i=2}^3 a_i^2(t, x) k_i \otimes k_i.$$

Meanwhile, we have

$$R_{01} = - \sum_{i=2}^3 a_i^2(t, x) k_i \otimes k_i + \delta R_{01}. \quad (5.1)$$

In next section, we will prove that δR_{01} is small.

5.2. The term II.

By (4.11) and (4.23), we have

$$2 \sum_{l \in Z^3} \beta_{1l}(t, x) b_{1l}(t, x) k_1 = \sum_{l \in Z^3} \alpha_l^2(\mu_1 t, \mu_1 x) c_1(t, x) k_1 = c_1(t, x) k_1.$$

By (4.3),

$$f_{0\ell} + 2 \sum_{l \in Z^3} \beta_{1l}(x, t) b_{1l}(x, t) k_1 = -c_2(t, x) k_2.$$

Meanwhile, we have

$$f_{01}(t, x) = -c_2(t, x) k_2 + \delta f_{01}. \quad (5.2)$$

Again in next section, we will prove that δf_{01} is small.

6. ESTIMATES ON δR_{01} AND δf_{01}

In the subsequent estimates, unless otherwise stated, C_0 always denotes an absolute constant which only depends on $r, \|v_0\|_0$ and C_m will in addition to depend on m and both them can vary from line to line.

In the following, we frequently use the elementary inequality

$$[fg]_m \leq C_m ([f]_m \|g\|_0 + [g]_m \|f\|_0) \quad (6.1)$$

for any $m \geq 0$.

Moreover, by the standard estimates on convolution,

$$\|c_{i\ell}\|_{C_{t,x}^m} + \|R_{0\ell}\|_{C_{t,x}^m} \leq C_m \Lambda \ell^{1-m} \quad \text{for any } m \geq 1, i = 1, 2, 3. \quad (6.2)$$

$$\|R_{0\ell} - R_0\|_0 + \|f_{0\ell} - f_0\|_0 \leq C_0 \Lambda \ell. \quad (6.3)$$

Then, we collect a classical estimates on the Hölder norms of compositions, their proof can be found in [18]:

Let $u : R^n \rightarrow R^N$ and $\Psi : R^N \rightarrow R$ be two smooth functions. Then, for every $m \in \mathbb{N} \setminus 0$ there is a constant $C_m = C_m(N, n)$ such that

$$[\Psi(u)]_m \leq C_m \sum_{i=1}^m [\Psi]_i [u]_1^{(i-1)\frac{m}{m-1}} [u]_m^{\frac{m-i}{m-1}}. \quad (6.4)$$

In particular,

$$[\Psi(u)]_1 \leq C_1 [\Psi]_1 [u]_1.$$

We summarize the main estimates on b_{1l} and β_{1l} in the following lemma.

Lemma 6.1. *For any $l \in Z^3$ and integer $m \geq 1$, we have*

$$\|b_{1l}\|_m + \|\beta_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^m + \mu_1 \ell^{-(m-1)}), \quad (6.5)$$

$$\|\partial_t b_{1l}\|_m + \|\partial_t \beta_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^{m+1} + \mu_1 \ell^{-m}), \quad (6.6)$$

$$\|\partial_{tt} b_{1l}\|_m + \|\partial_{tt} \beta_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^{m+2} + \mu_1 \ell^{-(m-1)}), \quad (6.7)$$

$$\|k_{\pm 1l}\|_m + \|h_{\pm 1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^m + \mu_1 \ell^{-(m-1)}), \quad (6.8)$$

$$\|\partial_t k_{\pm 1l}\|_m + \|\partial_t h_{\pm 1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^{m+1} + \mu_1 \ell^{-m}), \quad (6.9)$$

$$\|\partial_{tt} k_{\pm 1l}\|_m + \|\partial_{tt} h_{\pm 1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^{m+2} + \mu_1 \ell^{-(m-1)}) \quad (6.10)$$

and

$$\|b_{1l}\|_0 + \|\beta_{1l}\|_0 \leq C_0 \sqrt{\delta}, \quad (6.11)$$

$$\|\partial_t b_{1l}\|_0 + \|\partial_t \beta_{1l}\|_0 \leq C_0 \sqrt{\delta} \mu_1, \quad (6.12)$$

$$\|\partial_{tt} b_{1l}\|_0 + \|\partial_{tt} \beta_{1l}\|_0 \leq C_0 \sqrt{\delta} (\mu_1^2 + \mu_1 \ell^{-1}), \quad (6.13)$$

$$\|k_{\pm 1l}\|_0 + \|h_{\pm 1l}\|_0 \leq C_0 \sqrt{\delta}, \quad (6.14)$$

$$\|\partial_t k_{\pm 1l}\|_0 + \|\partial_t h_{\pm 1l}\|_0 \leq C_0 \sqrt{\delta} \mu_1, \quad (6.15)$$

$$\|\partial_{tt} k_{\pm 1l}\|_0 + \|\partial_{tt} h_{\pm 1l}\|_0 \leq C_0 \sqrt{\delta} (\mu_1^2 + \mu_1 \ell^{-1}). \quad (6.16)$$

Proof. First, notice the fact $\{(t, x) | \nabla e \neq 0\} \cap \{(t, x) | R_{0\ell} \neq 0\} = \emptyset$, we have, for any positive integer i ,

$$\nabla^i \left(\frac{R_{0\ell}}{e} \right) = \frac{\nabla^i (R_{0\ell})}{e},$$

thus for $m \geq 1$, by (4.6), (4.7), (6.2), (6.4) and parameter assumption (4.2)

$$\begin{aligned} \left[\gamma_{k_1} \left(Id - \frac{R_{0\ell}}{e} \right) (t, \cdot) \right]_m &\leq C_m \sum_{i=1}^m \|\nabla^i \gamma_{k_1}\|_0 \left[Id - \frac{R_{0\ell}}{e} (t, \cdot) \right]_1^{(i-1) \frac{m}{m-1}} \left[Id - \frac{R_{0\ell}}{e} (t, \cdot) \right]_m^{\frac{m-i}{m-1}} \\ &\leq C_m (\mu_1^m + \mu_1 \ell^{-(m-1)}). \end{aligned}$$

It's obvious that

$$\left\| \gamma_{k_1} \left(Id - \frac{R_{0\ell}}{e} \right) \right\|_0 \leq C_0.$$

Moreover, by (4.6) and parameter assumption (4.2), for any integer m ,

$$\|\rho\|_{C_{t,x}^m} \leq C_m \delta^{-\frac{1}{2}-(m-1)} \leq C_m \sqrt{\delta} \mu_1^m.$$

Then, recalling that

$$b_{1l}(t, x) := \frac{\rho(t, x)}{\sqrt{2}} \gamma_{k_1} \left(Id - \frac{R_{0\ell}}{e} \right) (t, x) \alpha_l(\mu_1 t, \mu_1 x),$$

thus, for $m \geq 1$, by (6.1) and parameter assumption (4.2), it's easy to get

$$\sup_t [b_{1l}(t, \cdot)]_m \leq C_m \sqrt{\delta} (\mu_1^m + \mu_1 \ell^{-(m-1)}).$$

By (6.2) and parameter assumption (4.2), for any integer $m \geq 1$,

$$\|c_{1\ell}\|_{C_{t,x}^m} \leq C_m \Lambda \ell^{-(m-1)} \leq C_m \delta \mu_1 \ell^{-(m-1)}.$$

By (4.5), $\|c_{1\ell}\|_0 \leq 2\delta$. By (4.6), (4.7), (6.4) and parameter assumption (4.2), for any integer m ,

$$\left\| \frac{1}{\sqrt{e}} \right\|_{C_{t,x}^m} \leq C_m \delta^{-\frac{1}{2}-m} \leq C_m \delta^{-\frac{1}{2}} \mu_1^m.$$

Notice that $\frac{1}{2} \leq \gamma_{k_1} \leq \frac{3}{2}$, then, by (4.23), (6.1) and parameter assumption (4.2), we also have

$$\|\beta_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^m + \mu_1 \ell^{-(m-1)}).$$

Thus, we complete the proof of (6.5). (6.11) is a direct result of (4.5) and (??).

By the definition (4.21) on $k_{\pm 1l}$, the definition (4.25) on $h_{\pm 1l}$, estimate (6.5) and parameter assumption (4.2), for $m \geq 1$, it's easy to obtain

$$\|k_{\pm 1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^m + \mu_1 \ell^{-(m-1)}), \quad \|h_{\pm 1l}\|_m \leq C_m \sqrt{\delta} (\mu_1^m + \mu_1 \ell^{-(m-1)}).$$

Thus, we complete the proof of (6.8). The case of $m = 0$ in (6.14) is also direct.

We introduce function

$$\Gamma(t, x) = \frac{\rho(t, x)}{\sqrt{2}} \gamma_{k_1} \left(Id - \frac{R_{0\ell}}{e} \right) (t, x).$$

By (4.6), (4.7) and parameter assumption (4.2) and notice the fact

$$\partial_t \left(\frac{R_{0\ell}}{e} \right) = \frac{(\partial_t R_0)_\ell}{e} - \frac{R_{0\ell} \partial_t e}{e^2} = \frac{(\partial_t R_0)_\ell}{e}, \quad \partial_{tt} \left(\frac{R_{0\ell}}{e} \right) = \frac{\partial_t R_0 * (\partial_t \varphi)_\ell \ell^{-1}}{e}$$

we have

$$\|\partial_t \Gamma\|_0 \leq C_0 \sqrt{\delta} \mu_1, \quad \|\partial_{tt} \Gamma\|_0 \leq C_0 \sqrt{\delta} (\mu_1^2 + \mu_1 \ell^{-1}).$$

Moreover, by (4.6), (6.1), (6.2), (6.4) and parameter assumption (4.2), for $m \geq 1$ we have

$$\|\partial_t \Gamma\|_m \leq C_m \sqrt{\delta} (\mu_1^{m+1} + \mu_1 \ell^{-m}), \quad \|\partial_{tt} \Gamma\|_m \leq C_m \sqrt{\delta} (\mu_1^{m+2} + \mu_1 \ell^{-m-1}).$$

Observe that

$$b_{1l}(t, x) = \Gamma(t, x) \alpha_l(\mu_1 t, \mu_1 x),$$

thus

$$\begin{aligned} \partial_t b_{1l} &= \partial_t \Gamma \alpha_l(\mu_1 t, \mu_1 x) + \mu_1 \Gamma (\partial_t \alpha)_l(\mu_1 t, \mu_1 x), \\ \partial_{tt} b_{1l} &= \partial_{tt} \Gamma \alpha_l(\mu_1 t, \mu_1 x) + 2\mu_1 \partial_t \Gamma (\partial_t \alpha)_l(\mu_1 t, \mu_1 x) + \mu_1^2 \Gamma (\partial_{tt} \alpha)_l(\mu_1 t, \mu_1 x). \end{aligned}$$

Hence, by (6.1) and the above estimate on Γ , we obtain

$$\|\partial_t b_{1l}\|_0 \leq C_0 \sqrt{\delta} \mu_1, \quad \|\partial_{tt} b_{1l}\|_0 \leq C_0 \sqrt{\delta} (\mu_1^2 + \mu_1 \ell^{-1}) \quad (6.17)$$

and

$$\|\partial_t b_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1 \ell^{-m} + \mu_1^{m+1}), \quad \|\partial_{tt} b_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1 \ell^{-m-1} + \mu_1^{m+2}).$$

The same argument gives

$$\begin{aligned} \|\partial_t \beta_{1l}\|_0 &\leq C_0 \sqrt{\delta} \mu_1, \quad \|\partial_{tt} \beta_{1l}\|_0 \leq C_0 \sqrt{\delta} (\mu_1^2 + \mu_1 \ell^{-1}), \\ \|\partial_t \beta_{1l}\|_m &\leq C_m \sqrt{\delta} (\mu_1 \ell^{-m} + \mu_1^{m+1}), \quad \|\partial_{tt} \beta_{1l}\|_m \leq C_m \sqrt{\delta} (\mu_1 \ell^{-m-1} + \mu_1^{m+2}). \end{aligned} \quad (6.18)$$

Thus, we obtain (6.6), (6.7), (6.12) and (6.13). Then, by the definition (4.21) on $k_{\pm 1l}$, the definition (4.25) on $h_{\pm 1l}$ and parameter assumption (4.2), it's easy to obtain (6.9), (6.10), (6.15) and (6.16). Thus, the proof of this lemma is complete. \square

Next, we give estimates on perturbations $w_{1o}, w_{1c}, \chi_{1o}, \chi_{1c}$.

Lemma 6.2 (Estimate on main perturbation and correction).

$$\|w_{1o}\|_0 \leq C_0\sqrt{\delta}, \quad \|w_{1o}\|_{C_{t,x}^1} \leq C_0\sqrt{\delta}\lambda_1, \quad \|\chi_{1o}\|_0 \leq C_0\sqrt{\delta}, \quad \|\chi_{1o}\|_{C_{t,x}^1} \leq C_0\sqrt{\delta}\lambda_1, \quad (6.19)$$

$$\|w_{1c}\|_0 \leq C_0\frac{\sqrt{\delta}\mu_1}{\lambda_1}, \quad \|w_{1c}\|_{C_{t,x}^1} \leq C_0\sqrt{\delta}\mu_1, \quad \|\chi_{1c}\|_0 \leq C_0\frac{\sqrt{\delta}\mu_1}{\lambda_1}, \quad \|\chi_{1c}\|_{C_{t,x}^1} \leq C_0\sqrt{\delta}\mu_1. \quad (6.20)$$

Proof. First, by (4.15) and (4.27), we know $\|w_{1o}\|_0 \leq C_0\sqrt{\delta}$, $\|\chi_{1o}\|_0 \leq C_0\sqrt{\delta}$. Since

$$\begin{aligned} \partial_t w_{1o} &= \sum_{l \in Z^3} \partial_t b_{1l} k_1 \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot \left((y,z) - v_0 \left(\frac{l}{\mu_1} \right) t \right)} + e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot \left((y,z) - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right) \\ &\quad - \sum_{l \in Z^3} b_{1l} i\lambda_1 2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) k_1 \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot \left((y,z) - v_0 \left(\frac{l}{\mu_1} \right) t \right)} - e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot \left((y,z) - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right), \end{aligned}$$

thus, by (6.11), (6.12) and parameter assumption (4.2), we obtain

$$\|\partial_t w_{1o}\|_0 \leq C_0\sqrt{\delta}\lambda_1.$$

The same argument gives

$$\|\nabla w_{1o}\|_0 \leq C_0\sqrt{\delta}\lambda_1, \quad \|\chi_{1o}\|_{C_{t,x}^1} \leq C_0\sqrt{\delta}\lambda_1.$$

Thus, we give a proof of (6.19).

Next, by (4.17), (6.5), parameter assumption (4.2), we get

$$\|w_{1cl}\|_0 \leq C_0\frac{\sqrt{\delta}\mu_1}{\lambda_1}.$$

Thus, from the property (4.1) on α_l , we arrive at

$$\|w_{1c}\|_0 \leq C_0\frac{\sqrt{\delta}\mu_1}{\lambda_1}.$$

Differentiating (4.17) in time

$$\begin{aligned} \partial_t w_{1cl} &= \sum_{l \in Z^3} \left\{ \frac{\nabla^\perp \partial_t b_{1l}}{i\lambda_1 2^{[l]}} \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} - e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right) \right. \\ &\quad - \nabla^\perp b_{1l} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} + e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right) \\ &\quad - \nabla^\perp \left(\frac{\nabla \partial_t b_{1l} \cdot k_1^\perp}{\lambda_1^2 2^{2[l]} |k_1|^2} \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} + e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right) \right. \\ &\quad \left. \left. - \nabla^\perp \left(\frac{\nabla b_{1l} \cdot k_1^\perp}{i\lambda_1 2^{[l]} |k_1|^2} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) \left(e^{i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} - e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot \left(x - v_0 \left(\frac{l}{\mu_1} \right) t \right)} \right) \right) \right\}. \end{aligned}$$

By (6.5), (6.6) and parameter assumption (4.2), we get

$$\|\partial_t w_{1c}\|_0 \leq C_0\sqrt{\delta}\mu_1.$$

Similarly, we have

$$\|\nabla w_{1c}\|_0 \leq C_0\sqrt{\delta}\mu_1, \quad \|\chi_{1c}\|_{C_{t,x}^1} \leq C_0\sqrt{\delta}\mu_1.$$

Collect the above estimates, we complete the proof of (6.20). \square

By (4.6), (4.7), (4.15), (4.27), (4.28) and lemma 6.2, it's easy to obtain the following estimate:

Corollary 6.3.

$$\begin{aligned} \|v_{01} - v_0\|_0 &\leq \frac{M\sqrt{\delta}}{6} + C_0 \frac{\sqrt{\delta}\mu_1}{\lambda_1}, \quad \|p_{01} - p_0\|_0 \leq M\delta, \quad \|\theta_{01} - \theta_0\|_0 \leq \frac{M\sqrt{\delta}}{4} + C_0 \frac{\sqrt{\delta}\mu_1}{\lambda_1}, \\ \|v_{01} - v_0\|_{C_{t,x}^1} &\leq C_0\lambda_1\sqrt{\delta}, \quad \|p_{01} - p_0\|_{C_{t,x}^1} \leq C_0, \quad \|\theta_{01} - \theta_0\|_{C_{t,x}^1} \leq C_0\lambda_1\sqrt{\delta}. \end{aligned} \quad (6.21)$$

6.1. Estimates on δR_{01} .

Recalling that

$$\begin{aligned} \delta R_{01} &= \mathcal{R}(\operatorname{div} M_1) + N_1 - \mathcal{R}(\chi_1 e_2) + \mathcal{R}\left\{\partial_t w_1 + \operatorname{div}\left[\sum_{l \in Z^3} \left(w_{1l} \otimes v_0\left(\frac{l}{\mu_1}\right) + v_0\left(\frac{l}{\mu_1}\right) \otimes w_{1l}\right)\right]\right\} \\ &\quad + (w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}). \end{aligned}$$

We split the stress into three parts:

(1) The oscillation part

$$\mathcal{R}(\operatorname{div} M_1) - \mathcal{R}(\chi_1 e_2).$$

(2) The transport part

$$\mathcal{R}\left\{\partial_t w_1 + \operatorname{div}\left[\sum_{l \in Z^3} \left(w_{1l} \otimes v_0\left(\frac{l}{\mu_1}\right) + v_0\left(\frac{l}{\mu_1}\right) \otimes w_{1l}\right)\right]\right\} = \mathcal{R}\left(\partial_t w_1 + \sum_{l \in Z^3} v_0\left(\frac{l}{\mu_1}\right) \cdot \nabla w_{1l}\right).$$

(3) The error part

$$N_1 + (w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}).$$

In the following we will estimate each term separately.

Lemma 6.4 (The oscillation part).

$$\|\mathcal{R}(\operatorname{div} M_1)\|_0 \leq C_0(\varepsilon) \frac{\delta\mu_1}{\lambda_1}, \quad \|\mathcal{R}(\operatorname{div} M_1)\|_{C_{t,x}^1} \leq C_0(\varepsilon) \delta\mu_1. \quad (6.22)$$

$$\|\mathcal{R}(\chi_1 e_2)\|_0 \leq C_0(\varepsilon) \frac{\sqrt{\delta}}{\lambda_1}, \quad \|\mathcal{R}(\chi_1 e_2)\|_{C_{t,x}^1} \leq C_0(\varepsilon) \sqrt{\delta}. \quad (6.23)$$

Proof. First, we have

$$M_1 = \sum_{j=0}^7 \sum_{[l]=j} k_1 \otimes k_1 \left(e^{2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) b_{1l}^2 + \sum_{l, l' \in Z^3, l \neq l'} w_{1ol} \otimes w_{1ol'}.$$

Since $k_1 \cdot k_1^\perp = 0$, then

$$\operatorname{div} M_1 = M_{11} + M_{12}.$$

where

$$\begin{aligned} M_{11} &= \sum_{j=0}^7 \sum_{[l]=j} k_1 \otimes k_1 \nabla(b_{1l}^2) \left(e^{2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right), \\ M_{12} &= \sum_{l, l' \in Z^3, l \neq l'} \operatorname{div}(w_{1ol} \otimes w_{1ol'}). \end{aligned}$$

By (6.5), (6.11), Proposition 3.1 with $m = \left[1 + \frac{1}{\varepsilon}\right] + 1$ and parameter assumption (4.2), we have

$$\begin{aligned} \|\mathcal{R}(M_{11})\|_0 &\leq C_m \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\|\sum_{[l]=j} \nabla(b_{1l}^2)\|_i}{(\lambda_1 2^j)^{i+1}} + \frac{\|\sum_{[l]=j} \nabla(b_{1l}^2)\|_m}{(\lambda_1 2^j)^m} \right) \\ &\leq C_m \sum_{j=0}^7 \delta \left(\frac{\mu_1}{\lambda_1 2^j} + \frac{\mu_1^{m+1} + \mu_1 \ell^{-m}}{(\lambda_1 2^j)^m} \right) \leq C_m \frac{\delta \mu_1}{\lambda_1}. \end{aligned}$$

where we use the fact: $b_{1l}b_{1l'} = 0$ if $|l - l'| \geq 2$.

Notice

$$\begin{aligned} M_{12} &= \sum_{j=0}^7 \sum_{[l]=j} \sum_{l' \in Z^3, 1 \leq |l' - l| < 2} k_1 \otimes k_1 \nabla(b_{1l}b_{1l'}) \left(e^{i\lambda_1(2^j + 2^{[l']})k_1^\perp \cdot x - i g_{1,l,l'}(t)} + e^{i\lambda_1(2^j - 2^{[l']})k_1^\perp \cdot x - i \bar{g}_{1,l,l'}(t)} \right. \\ &\quad \left. + e^{i\lambda_1(2^{[l']} - 2^j)k_1^\perp \cdot x + i \bar{g}_{1,l,l'}(t)} + e^{-i\lambda_1(2^j + 2^{[l']})k_1^\perp \cdot x + i g_{1,l,l'}(t)} \right), \end{aligned}$$

where

$$g_{1,l,l'}(t) = \lambda_1 \left(2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) t + 2^{[l']} k_1^\perp \cdot v_0 \left(\frac{l'}{\mu_1} \right) t \right), \quad \bar{g}_{1,l,l'}(t) = \lambda_1 \left(2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) - 2^{[l']} k_1^\perp \cdot v_0 \left(\frac{l'}{\mu_1} \right) \right).$$

Following the same strategy as M_{11} , we deduce

$$\begin{aligned} \|\mathcal{R}(M_{12})\|_0 &\leq C_m \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\left\| \sum_{[l]=j} \sum_{l' \in Z^3, 1 \leq |l' - l| < 2} \nabla(b_{1l}b_{1l'}) \right\|_i}{\lambda_1^{i+1}} + \frac{\left\| \sum_{[l]=j} \sum_{l' \in Z^3, 1 \leq |l' - l| < 2} \nabla(b_{1l}b_{1l'}) \right\|_m}{\lambda_1^m} \right) \\ &\leq C_m \sum_{j=0}^7 \delta \left(\frac{\mu_1}{\lambda_1} + \frac{\mu_1^{m+1} + \mu_1 \ell^{-m}}{\lambda_1^m} \right) \leq C_m \frac{\delta \mu_1}{\lambda_1}. \end{aligned} \quad (6.24)$$

Thus, the first estimate of (6.22) follows easily. A direct calculation gives

$$\begin{aligned} \partial_t M_{11} &= \sum_{j=0}^7 \sum_{[l]=j} k_1 \otimes k_1 \nabla \partial_t(b_{1l}^2) \left(e^{2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) \\ &\quad - \sum_{j=0}^7 \sum_{[l]=j} k_1 \otimes k_1 \nabla(b_{1l}^2) 2i\lambda_1 2^j k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) \left(e^{2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} - e^{-2i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right). \end{aligned}$$

Thus, by (6.5), (6.6), (6.11), Proposition 3.1 with $m = \left[1 + \frac{1}{\varepsilon}\right] + 1$ and parameter assumption (4.2), we have

$$\begin{aligned} \|\partial_t \mathcal{R}(M_{11})\|_0 &\leq C_m \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\|\sum_{[l]=j} \nabla \partial_t(b_{1l}^2)\|_i}{(\lambda_1 2^j)^{i+1}} + \frac{\|\sum_{[l]=j} \nabla \partial_t(b_{1l}^2)\|_m}{(\lambda_1 2^j)^m} \right) \\ &\quad + C_m \lambda_1 \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\|\sum_{[l]=j} \nabla(b_{1l}^2)\|_i}{(\lambda_1 2^j)^{i+1}} + \frac{\|\sum_{[l]=j} \nabla(b_{1l}^2)\|_m}{(\lambda_1 2^j)^m} \right) \\ &\leq C_m \lambda_1 \sum_{j=0}^7 \delta \left(\frac{\mu_1}{\lambda_1 2^j} + \frac{\mu_1^{m+1} + \mu_1 \ell^{-m}}{(\lambda_1 2^j)^m} \right) \leq C_m \delta \mu_1. \end{aligned}$$

By the same argument, we have

$$\|\partial_t \mathcal{R}(M_{12})\|_0 \leq C_m \delta \mu_1, \quad \|\nabla \mathcal{R}(M_{11})\|_0 \leq C_m \delta \mu_1, \quad \|\nabla \mathcal{R}(M_{12})\|_0 \leq C_m \delta \mu_1.$$

Putting these estimates together, we obtain the second estimate of (6.22).
Recalling (4.26)

$$\chi_1 = \sum_{j=0}^7 \sum_{[l]=j} \left(h_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + h_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right).$$

By (6.8), (6.9), (6.14), (6.15) and using a similar argument as above, we obtain

$$\|\mathcal{R}(\chi_1 e_2)\|_0 \leq C_m \frac{\sqrt{\delta}}{\lambda_1}, \quad \|\mathcal{R}(\chi_1 e_2)\|_{C_{t,x}^1} \leq C_m \sqrt{\delta}.$$

Thus, the proof of this lemma is complete. \square

Lemma 6.5 (The transport part).

$$\begin{aligned} \left\| \mathcal{R} \left(\partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} \right) \right\|_0 &\leq C_0(\varepsilon) \frac{\sqrt{\delta} \mu_1}{\lambda_1}, \\ \left\| \mathcal{R} \left(\partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} \right) \right\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \sqrt{\delta} \mu_1. \end{aligned} \quad (6.25)$$

Proof. Recalling (4.22)

$$w_1 = \sum_{j=0}^7 \sum_{[l]=j} \left(k_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + k_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right).$$

Thus, using the identity

$$\left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) e^{\pm i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} = 0,$$

we deduce

$$\begin{aligned} \partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} &= \sum_{j=0}^7 \sum_{[l]=j} \left(\left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right. \\ &\quad \left. + \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right). \end{aligned}$$

By Proposition 3.1 with $m = \left[1 + \frac{1}{\varepsilon} \right] + 1$, (6.8), (6.9), (6.15) and parameter assumption (4.2), we have

$$\begin{aligned} &\left\| \mathcal{R} \left(\partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} \right) \right\|_0 \\ &\leq C_m \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\left\| \sum_{[l]=j} \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{1l} \right\|_i}{(\lambda_1 2^j)^{i+1}} + \frac{\left\| \sum_{[l]=j} \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{1l} \right\|_m}{(\lambda_1 2^j)^m} \right) \\ &\quad + C_m \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\left\| \sum_{[l]=j} \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{-1l} \right\|_i}{(\lambda_1 2^j)^{i+1}} + \frac{\left\| \sum_{[l]=j} \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{-1l} \right\|_m}{(\lambda_1 2^j)^m} \right) \\ &\leq C_m \sum_{j=0}^7 \sqrt{\delta} \left(\frac{\mu_1}{\lambda_1 2^j} + \frac{\mu_1^{m+1} + \mu_1 \ell^{-m}}{(\lambda_1 2^j)^m} \right) \leq C_m \frac{\sqrt{\delta} \mu_1}{\lambda_1}. \end{aligned} \quad (6.26)$$

A direct calculation gives

$$\begin{aligned} \partial_t \left(\partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} \right) &= \sum_{j=0}^7 \sum_{[l]=j} \left(\left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) \partial_t k_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right. \\ &\quad + \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) \partial_t k_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ &\quad - \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{1l} i\lambda_1 2^j k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ &\quad \left. + \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) k_{-1l} i\lambda_1 2^j k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right), \end{aligned}$$

thus, by (6.8)-(6.10), (6.15)-(6.16) and applying the same argument as above, we arrive at

$$\left\| \partial_t \mathcal{R} \left(\partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} \right) \right\|_0 \leq C_m \sqrt{\delta} \mu_1.$$

Similarly, we have

$$\left\| \nabla \mathcal{R} \left(\partial_t w_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{1l} \right) \right\|_0 \leq C_m \sqrt{\delta} \mu_1.$$

Then we proved the Lemma 6.5. \square

Lemma 6.6 (Estimates on error part I).

$$\|N_1\|_0 \leq C_0 \left(\sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda \ell \right), \quad \|N_1\|_{C_{t,x}^1} \leq C_0 \lambda_1 \left(\sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda \ell \right). \quad (6.27)$$

Proof. We may rewrite N_1 as

$$N_1 = N_{11} + N_{12},$$

where

$$N_{11} = \sum_{l \in Z^3} \left[w_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) + \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \otimes w_{1l} \right], \quad N_{12} = R_0 - R_{0\ell}.$$

For the term N_{11} , by (4.22), we have

$$\begin{aligned} &\sum_{l \in Z^3} w_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \\ &= \sum_{l \in Z^3} \left(k_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) e^{i\lambda_1 2^{|l|} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + k_{-1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right). \end{aligned}$$

Obviously, $k_{1l}(x, t) \neq 0$ implies $|(\mu_1 t, \mu_1 x) - l| \leq 1$, therefore, by (6.14)

$$\left| k_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \right| \leq C_0 \sqrt{\delta} \frac{\|\nabla v_0\|_0}{\mu_1} \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}.$$

By (4.1), it's easy to get

$$\left\| \sum_{l \in Z^3} k_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}.$$

Similarly,

$$\left\| \sum_{l \in Z^3} k_{-1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}.$$

Thus,

$$\left\| \sum_{l \in Z^3} w_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}. \quad (6.28)$$

Following the same strategy:

$$\left\| \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \otimes w_{1l} \right\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}. \quad (6.29)$$

Finally, putting (6.28) and (6.29) together, we arrive at

$$\|N_{11}\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}. \quad (6.30)$$

Moreover, we have

$$\begin{aligned} & \partial_t \left(\sum_{l \in Z^3} w_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \right) \\ &= \sum_{l \in Z^3} \left(\partial_t k_{1l} e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + \partial_t k_{-1l} e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \\ &+ \sum_{l \in Z^3} \left(k_{1l} e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + k_{-1l} e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) \otimes \partial_t v_0 \\ &+ \sum_{l \in Z^3} \left(-k_{1l} i\lambda_1 2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right. \\ &\quad \left. + k_{-1l} i\lambda_1 2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right), \end{aligned}$$

thus, by (6.14), (6.15) and parameter assumption(4.2)

$$\left\| \partial_t \left(\sum_{l \in Z^3} w_{1l} \otimes \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \right) \right\|_0 \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}.$$

Similarly, we have

$$\left\| \partial_t \left(\sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \otimes w_{1l} \right) \right\|_0 \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}.$$

Therefore,

$$\|\partial_t N_{11}\|_0 \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}. \quad (6.31)$$

By a similar argument, we have

$$\|\nabla N_{11}\|_0 \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}. \quad (6.32)$$

By (6.3), we have

$$\|R_0 - R_{0\ell}\|_0 \leq C_0 \Lambda \ell, \quad \|\partial_t(R_0 - R_{0\ell})\|_0 \leq C_0 \Lambda, \quad \|\nabla(R_0 - R_{0,\ell})\|_0 \leq C_0 \Lambda.$$

Thus, by parameter assumption(4.2), we arrive at

$$\|N_{12}\|_0 \leq C_0 \Lambda \ell, \quad \|N_{12}\|_{C_{t,x}^1} \leq C_0 \lambda_1 \Lambda \ell. \quad (6.33)$$

Collecting (6.30), (6.31), (6.32) and (6.33), we obtain

$$\|N_1\|_0 \leq C_0 \left(\sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda \ell \right), \quad \|N_1\|_{C_{t,x}^1} \leq C_0 \lambda_1 \left(\sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda \ell \right).$$

We complete our proof of this lemma. \square

Lemma 6.7 (Estimates on error part II).

$$\begin{aligned} \|w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}\|_0 &\leq C_0 \frac{\delta\mu_1}{\lambda_1}, \\ \|w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}\|_{C_{t,x}^1} &\leq C_0 \delta\mu_1. \end{aligned} \quad (6.34)$$

Proof. By (6.19) and (6.20), we have

$$\|w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}\|_0 \leq C_0(\|w_{1o}\|_0\|w_{1c}\|_0 + \|w_{1c}\|_0^2) \leq C_0 \frac{\delta\mu_1}{\lambda_1}$$

and

$$\|w_{1o} \otimes w_{1c} + w_{1c} \otimes w_{1o} + w_{1c} \otimes w_{1c}\|_{C_{t,x}^1} \leq C_0(\|w_{1o}\|_0\|w_{1c}\|_{C_{t,x}^1} + \|w_{1o}\|_{C_{t,x}^1}\|w_{1c}\|_0) \leq C_0 \delta\mu_1.$$

thus, we complete the proof of this lemma. \square

Finally, from Lemma 6.4, Lemma 6.5, Lemma 6.6 and Lemma 6.7, we conclude

$$\|\delta R_{01}\|_0 \leq C_0(\varepsilon) \left(\frac{\sqrt{\delta}\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right), \quad \|\delta R_{01}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_1 \left(\frac{\sqrt{\delta}\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right). \quad (6.35)$$

6.2. Estimates on δf_{01} .

Recalling that

$$\begin{aligned} \delta f_{01} = & \mathcal{G}(\operatorname{div} K_1) + \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) + w_{1o} \chi_{1c} + f_0 - f_{0\ell} \\ & + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} + w_{1c} \chi_1 + \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_1} \right) \right). \end{aligned}$$

As before, we split δf_{01} into three parts:

(1) The oscillation part:

$$\mathcal{G}(\operatorname{div} K_1).$$

(2) The transport part:

$$\mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right).$$

(3) The error part:

$$w_{1c} \chi_1 + w_{1o} \chi_{1c} + f_0 - f_{0\ell} + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} + \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_1} \right) \right).$$

Lemma 6.8 (The oscillation part).

$$\|\mathcal{G}(\operatorname{div} K_1)\|_0 \leq C_0(\varepsilon) \frac{\delta\mu_1}{\lambda_1}, \quad \|\mathcal{G}(\operatorname{div} K_1)\|_{C_{t,x}^1} \leq C_0(\varepsilon) \delta\mu_1. \quad (6.36)$$

Proof. Recalling the notations of K_1 and $[l]$, we have

$$K_1 = \sum_{j=0}^7 \sum_{[l]=j} \beta_{1l} b_{1l} k_1 \left(e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) + \sum_{l, l' \in Z^3, l \neq l'} w_{1ol} \chi_{1ol'},$$

thus

$$\operatorname{div} K_1 = K_{11} + K_{12},$$

where

$$K_{11} = \sum_{j=0}^7 \sum_{[l]=j} \nabla(\beta_{1l} b_{1l}) \cdot k_1 \left(e^{i\lambda_1 2^{|l|} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right),$$

$$K_{12} = \sum_{l, l' \in Z^3, l \neq l'} \operatorname{div}(w_{10l} \chi_{10l'}).$$

By (6.5), (6.11), Proposition (3.2) with $m = \left[1 + \frac{1}{\varepsilon}\right] + 1$ and parameter assumption (4.2), we have

$$\begin{aligned} \|\mathcal{R}(K_{11})\|_0 &\leq C_m \sum_{j=0}^7 \left(\sum_{i=0}^{m-1} \frac{\|\sum_{[l]=j} \nabla(\beta_{1l} b_{1l})\|_i}{(\lambda_1 2^j)^{i+1}} + \frac{\|\sum_{[l]=j} \nabla(\beta_{1l} b_{1l})\|_m}{(\lambda_1 2^j)^m} \right) \\ &\leq C_m \sum_{j=0}^7 \delta \left(\frac{\mu_1}{\lambda_1 2^j} + \frac{\mu_1^{m+1} + \mu_1 \ell^{-m}}{(\lambda_1 2^j)^m} \right) \leq C_m \frac{\delta \mu_1}{\lambda_1}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} K_{12} &= \sum_{j=0}^7 \sum_{[l]=j} \sum_{l' \in Z^3, 1 \leq |l' - l| < 2} k_1 \cdot \nabla(b_{1l} \beta_{1l'}) \left(e^{i\lambda_1 (2^j + 2^{|l'|}) k_1^\perp \cdot x - i g_{1,l,l'}(t)} + e^{i\lambda_1 (2^j - 2^{|l'|}) k_1^\perp \cdot x - i \bar{g}_{1,l,l'}(t)} \right. \\ &\quad \left. + e^{i\lambda_1 (2^{|l'|} - 2^j) k_1^\perp \cdot x + i \bar{g}_{1,l,l'}(t)} + e^{-i\lambda_1 (2^j + 2^{|l'|}) k_1^\perp \cdot x + i g_{1,l,l'}(t)} \right), \end{aligned}$$

as estimate (6.24) on M_{12} , by (6.5), (6.11) and Proposition (3.2), we have

$$\|\mathcal{G}(K_{12})\|_0 \leq C_m \frac{\delta \mu_1}{\lambda_1}.$$

A straightforward computation gives

$$\begin{aligned} \partial_t K_{11} &= \sum_{j=0}^7 \sum_{[l]=j} \nabla \partial_t(\beta_{1l} b_{1l}) \cdot k_1 \left(e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right) \\ &\quad - \sum_{j=0}^7 \sum_{[l]=j} \nabla(\beta_{1l} b_{1l}) \cdot k_1 i\lambda_1 2^j k_1^\perp \cdot v_0\left(\frac{l}{\mu_1}\right) \left(e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} - e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right), \end{aligned}$$

thus, by the same argument

$$\|\partial_t \mathcal{G}(K_{11})\|_0 \leq C_m \delta \mu_1.$$

Similarly,

$$\|\partial_t \mathcal{G}(K_{12})\|_0 \leq C_m \delta \mu_1, \quad \|\nabla \mathcal{G}(K_{11})\|_0 \leq C_m \delta \mu_1, \quad \|\nabla \mathcal{G}(K_{12})\|_0 \leq C_m \delta \mu_1.$$

We complete the proof of this lemma. \square

Lemma 6.9 (The transport part).

$$\begin{aligned} \left\| \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) \right\|_0 &\leq C_0(\varepsilon) \frac{\sqrt{\delta} \mu_1}{\lambda_1}, \\ \left\| \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) \right\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \sqrt{\delta} \mu_1. \end{aligned} \tag{6.37}$$

Proof. Recalling the notation of χ_1 , we have

$$\begin{aligned} & \partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \\ &= \sum_{j=0}^7 \sum_{[l]=j} \left\{ \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) h_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) h_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right\}. \end{aligned}$$

By Proposition 3.2 with $m = \left[1 + \frac{1}{\varepsilon} \right] + 1$, (6.8), (6.9), (6.15) and parameter assumption (4.2), as estimate in (6.26), we have

$$\left\| \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) \right\|_0 \leq C_m \sum_{j=0}^7 \sqrt{\delta} \left(\frac{\mu_1}{\lambda_1 2^j} + \frac{\mu_1^{m+1} + \mu_1 \ell^{-m}}{(\lambda_1 2^j)^m} \right) \leq C_m \frac{\sqrt{\delta} \mu_1}{\lambda_1}.$$

And since

$$\begin{aligned} \partial_t \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) &= \sum_{j=0}^7 \sum_{[l]=j} \left(\left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) \partial_t h_{1l} e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right. \\ &\quad + \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) \partial_t h_{-1l} e^{-i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ &\quad - \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) h_{1l} i\lambda_1 2^j k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ &\quad \left. + \left(\partial_t + v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \right) h_{-1l} i\lambda_1 2^j k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{i\lambda_1 2^j k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \right), \end{aligned}$$

then, by the same argument, we obtain

$$\left\| \partial_t \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) \right\|_0 \leq C_m \sqrt{\delta} \mu_1.$$

Similarly, we have

$$\left\| \nabla \mathcal{G} \left(\partial_t \chi_1 + \sum_{l \in Z^3} v_0 \left(\frac{l}{\mu_1} \right) \cdot \nabla \chi_{1l} \right) \right\|_0 \leq C_m \sqrt{\delta} \mu_1.$$

Then we complete our proof of this Lemma. \square

Lemma 6.10 (The error part).

$$\begin{aligned} & \left\| w_{1c} \chi_1 + w_{1o} \chi_{1c} + f_0 - f_{0\ell} + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} + \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_0 \\ & \leq C_0 \left(\frac{\delta \mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda \ell \right), \\ & \left\| w_{1c} \chi_1 + w_{1o} \chi_{1c} + f_0 - f_{0\ell} + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} + \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_{C_{t,x}^1} \\ & \leq C_0 \lambda_1 \left(\frac{\delta \mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda \ell \right). \end{aligned} \tag{6.38}$$

Proof. Using Lemma 6.2, it's easy to obtain

$$\|w_{1c} \chi_1\|_0 \leq C_0 \frac{\delta \mu_1}{\lambda_1}, \quad \|w_{1o} \chi_{1c}\|_0 \leq C_0 \frac{\delta \mu_1}{\lambda_1}, \quad \|w_{1c} \chi_1\|_{C_{t,x}^1} \leq C_0 \delta \mu_1, \quad \|w_{1o} \chi_{1c}\|_{C_{t,x}^1} \leq C_0 \delta \mu_1.$$

Then,

$$\begin{aligned} & \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} \\ &= \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) h_{1l} e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) h_{-1l} e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}. \end{aligned}$$

Obviously, $h_{1l}(x, t), h_{-1l} \neq 0$ implies $|(\mu_1 t, \mu_1 x) - l| \leq 1$. Therefore, following the same strategy as estimate (6.28), we obtain

$$\left\| \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} \right\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}.$$

Similarly, we have

$$\left\| \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_0 \leq C_0 \sqrt{\delta} \frac{\Lambda}{\mu_1}.$$

By calculation we have

$$\begin{aligned} & \partial_t \left(\sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} \right) \\ &= \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \partial_t h_{1l} e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \partial_t h_{-1l} e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ & \quad - \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) h_{1l} i\lambda_1 2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ & \quad + \sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) h_{-1l} i\lambda_1 2^{[l]} k_1^\perp \cdot v_0 \left(\frac{l}{\mu_1} \right) e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} \\ & \quad + \sum_{l \in Z^3} \partial_t v_0 h_{1l} e^{i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)} + \sum_{l \in Z^3} \partial_t v_0 h_{-1l} e^{-i\lambda_1 2^{[l]} k_1^\perp \cdot (x - v_0(\frac{l}{\mu_1})t)}. \end{aligned}$$

Therefore, by (6.14), (6.15) and parameter assumption(4.2)

$$\left\| \partial_t \left(\sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} \right) \right\|_0 \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}.$$

Similarly, we have

$$\left\| \nabla \left(\sum_{l \in Z^3} \left(v_0 - v_0 \left(\frac{l}{\mu_1} \right) \right) \chi_{1l} \right) \right\|_0 \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}.$$

Applying the similar argument, we have

$$\left\| \sum_{l \in Z^3} w_{1l} \left(\theta_0 - \theta_0 \left(\frac{l}{\mu_1} \right) \right) \right\|_{C_{t,x}^1} \leq C_0 \sqrt{\delta} \lambda_1 \frac{\Lambda}{\mu_1}.$$

By (6.3)

$$\|f_0 - f_{0\ell}\|_0 \leq C_0 \Lambda \ell, \quad \|f_0 - f_{0\ell}\|_{C_{t,x}^1} \leq C_0 \Lambda.$$

Collecting the above estimates together, we complete our proof. \square

Combining lemma 6.8, lemma 6.9 and lemma 6.10, we conclude

$$\|\delta f_{01}\|_0 \leq C_0(\varepsilon) \left(\frac{\sqrt{\delta}\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right), \quad \|\delta f_{01}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_1 \left(\frac{\sqrt{\delta}\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right). \quad (6.39)$$

Finally, by (5.1), (5.2), Corollary 6.21, (6.35) and (6.39), we conclude that $(v_{01}, p_{01}, \theta_{01}, R_{01}, f_{01}) \in C_c^\infty(Q_{r+\delta})$ solves system (2.1) and satisfies

$$R_{01}(t, x) = - \sum_{i=2}^3 a_i^2(t, x) k_i \otimes k_i + \delta R_{01}, \quad f_{01}(t, x) := -c_2(t, x) k_2 + \delta f_{01}$$

with

$$\begin{aligned} \|v_{01} - v_0\|_0 &\leq \frac{M\sqrt{\delta}}{6} + C_0 \frac{\sqrt{\delta}\mu_1}{\lambda_1}, \quad \|p_{01} - p_0\|_0 \leq M\delta, \quad \|\theta_{01} - \theta_0\|_0 \leq \frac{M\sqrt{\delta}}{4} + C_0 \frac{\sqrt{\delta}\mu_1}{\lambda_1}, \\ \|v_{01} - v_0\|_{C_{t,x}^1} &\leq C_0 \lambda_1 \sqrt{\delta}, \quad \|p_{01} - p_0\|_{C_{t,x}^1} \leq C_0, \quad \|\theta_{01} - \theta_0\|_{C_{t,x}^1} \leq C_0 \lambda_1 \sqrt{\delta}, \\ \|\delta R_{01}\|_0 &\leq C_0(\varepsilon) \left(\sqrt{\delta} \frac{\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right), \quad \|\delta R_{01}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_1 \left(\sqrt{\delta} \frac{\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right), \\ \|\delta f_{01}\|_0 &\leq C_0(\varepsilon) \left(\sqrt{\delta} \frac{\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right), \quad \|\delta f_{01}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_1 \left(\sqrt{\delta} \frac{\mu_1}{\lambda_1} + \sqrt{\delta} \frac{\Lambda}{\mu_1} + \Lambda\ell \right). \end{aligned}$$

Thus, we complete the first step.

7. CONSTRUCTIONS OF $(v_{0n}, p_{0n}, \theta_{0n}, R_{0n}, f_{0n})$, $2 \leq n \leq 3$

In this section, we suppose $2 \leq n \leq 3$ and construct $(v_{0n}, p_{0n}, \theta_{0n}, R_{0n}, f_{0n})$ by inductions.

Suppose that for $1 \leq m < n \leq 3$, $(v_{0m}, p_{0m}, \theta_{0m}, R_{0m}, f_{0m}) \in C_c^\infty(Q_{r+\delta})$ solves system (2.1) and satisfies

$$R_{0m} = - \sum_{i=m+1}^3 a_i^2 k_i \otimes k_i + \sum_{i=i}^m \delta R_{0i}, \quad f_{0m} := - \sum_{i=m+1}^3 c_i k_i + \sum_{i=i}^m \delta f_{0i} \quad (7.1)$$

with

$$\begin{aligned} \|v_{0m} - v_{0(m-1)}\|_0 &\leq \frac{M\sqrt{\delta}}{6} + C_0 \frac{\sqrt{\delta}\mu_m}{\lambda_m}, \quad \|v_{0m} - v_{0(m-1)}\|_{C_{t,x}^1} \leq C_0 \lambda_m \sqrt{\delta}, \\ \|p_{0m} - p_{0(m-1)}\|_0 &\leq \begin{cases} M\delta, m=1 \\ 0, m=2 \end{cases} \quad \|p_{0m} - p_{0(m-1)}\|_{C_{t,x}^1} \leq \begin{cases} C_0, m=1 \\ 0, m=2 \end{cases} \\ \|\theta_{0m} - \theta_{0(m-1)}\|_0 &\leq \frac{M\sqrt{\delta}}{6} + C_0 \frac{\sqrt{\delta}\mu_m}{\lambda_m}, \quad \|\theta_{0m} - \theta_{0(m-1)}\|_{C_{t,x}^1} \leq C_0 \lambda_m \sqrt{\delta}. \end{aligned} \quad (7.2)$$

and

$$\begin{aligned} \|\delta R_{0m}\|_0 &\leq C_0(\varepsilon) \left(\frac{\sqrt{\delta}\mu_m}{\lambda_m} + \sqrt{\delta} \frac{\sqrt{\delta}\lambda_{m-1}}{\mu_m} \right), \quad \|\delta f_{0m}\|_0 \leq C_0(\varepsilon) \left(\frac{\sqrt{\delta}\mu_m}{\lambda_m} + \sqrt{\delta} \frac{\sqrt{\delta}\lambda_{m-1}}{\mu_m} \right), \\ \|\delta R_{0m}\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \lambda_m \left(\frac{\sqrt{\delta}\mu_m}{\lambda_m} + \sqrt{\delta} \frac{\sqrt{\delta}\lambda_{m-1}}{\mu_m} \right), \quad \|\delta f_{0m}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_m \left(\frac{\sqrt{\delta}\mu_m}{\lambda_m} + \sqrt{\delta} \frac{\sqrt{\delta}\lambda_{m-1}}{\mu_m} \right). \end{aligned} \quad (7.3)$$

Here $(v_{00}, p_{00}, \theta_{00}) = (v_0, p_0, \theta_0)$ and the parameter μ_m, λ_m satisfies

$$\lambda_m \geq \max\{\mu_m^{1+\varepsilon}, \ell^{-(1+\varepsilon)}\}, \quad \mu_m > \mu_{m-1} \quad (7.4)$$

and $\lambda_0 = \Lambda\delta^{-\frac{1}{2}} + \frac{\mu_1\Lambda\ell}{\delta}, \mu_0 = 1$. Next, we perform the n-th step.

7.1. Construction of n-th perturbation on velocity.

7.1.1. *Main perturbation w_{no} .* For any $l \in Z^3$, we denote b_{nl} by

$$b_{nl}(t, x) := \frac{a_n(t, x)\alpha_l(\mu_n t, \mu_n x)}{\sqrt{2}} \quad (7.5)$$

and main l -perturbation w_{1ol} by

$$w_{nol} := b_{nl}k_n \left(e^{i\lambda_n 2^{[l]}k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + e^{-i\lambda_n 2^{[l]}k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right), \quad (7.6)$$

where two parameters μ_n and λ_n will be chosen with

$$\lambda_n \geq \max\{\mu_n^{1+\varepsilon}, \ell^{-(1+\varepsilon)}\}, \quad \mu_n > \mu_{n-1}. \quad (7.7)$$

Then we denote n -th main perturbation w_{no} by

$$w_{no} := \sum_{l \in Z^3} w_{nol} = \sum_{j=0}^7 \sum_{[l]=j} b_{nl}k_n \left(e^{i\lambda_n 2^j k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + e^{-i\lambda_n 2^j k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right).$$

Obviously, w_{nol}, w_{no} are all real 2-dimensional vector-valued functions.

By (4.1), $\text{supp}\alpha_l \cap \text{supp}\alpha_{l'} = \emptyset$ if $|l - l'| \geq 2$, thus the above summation is finite and

$$\|w_{no}\|_0 \leq \frac{M\sqrt{\delta}}{6}. \quad (7.8)$$

Moreover, since $a_n(t, x) \in C_c^\infty(Q_{r+\delta})$, we know that for any $l \in Z^3$,

$$b_{nl} \in C_c^\infty(Q_{r+\delta}), \quad w_{nol} \in C_c^\infty(Q_{r+\delta}) \quad w_{no} \in C_c^\infty(Q_{r+\delta}). \quad (7.9)$$

7.1.2. *The correction w_{nc} and the n -th perturbation w_n .* We denote the l -correction w_{ncl} by

$$w_{ncl} := \frac{\nabla^\perp b_{nl}}{i\lambda_n 2^{[l]}} \left(e^{i\lambda_n 2^{[l]}k_1^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} - e^{-i\lambda_n 2^{[l]}k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right) \\ - \nabla^\perp \left(\frac{\nabla b_{nl} \cdot k_n^\perp}{\lambda_n^2 2^{2[l]}|k_n|^2} \left(e^{i\lambda_n 2^{[l]}k_1^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + e^{-i\lambda_n 2^{[l]}k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right) \right).$$

Then the n -th correction is given by

$$w_{nc} := \sum_{l \in Z^3} w_{ncl}.$$

Finally, we denote n -th perturbation w_n by

$$w_n := w_{no} + w_{nc}.$$

Thus, if we set

$$w_{nl} := w_{nol} + w_{ncl},$$

then

$$w_{nl} = \nabla^\perp \text{div} \left(- \frac{b_{nl}}{\lambda_n^2 2^{2[l]}|k_n|^2} k_n^\perp 2 \cos \left(\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t) \right) \right)$$

and

$$w_n = \sum_{l \in Z^3} w_{nl}, \quad \text{div} w_{nl} = 0, \quad \int_{R^2} w_{nl} dx = 0, \quad \int_{R^2} (x_i w_{nlj} - x_j w_{nli}) dx = 0, \quad i, j = 1, 2.$$

In fact

$$\int_{R^2} (x_1 w_{nl2} - x_2 w_{nl1}) dx = \int_{R^2} (x_1 \partial_1 \text{div} \vec{a}_n + x_2 \partial_2 \text{div} \vec{a}_n) dx = -2 \int_{R^2} \text{div} \vec{a}_n dx = 0,$$

where

$$\vec{a}_n = - \frac{b_{nl}}{\lambda_n^2 2^{2[l]}|k_n|^2} k_1^\perp 2 \cos \left(\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t) \right)$$

is a vector-valued smooth function with compact support. Obviously, we have

$$\operatorname{div} w_n = 0.$$

Moreover, if we set

$$\begin{aligned} k_{nl} &:= b_{nl} k_n + \frac{\nabla^\perp b_{nl}}{i\lambda_n 2^{[l]}} - \nabla^\perp \left(\frac{\nabla b_{nl} \cdot k_n^\perp}{\lambda_n^2 2^{2[l]} |k_n|^2} \right) + \frac{\nabla b_{nl} \cdot k_n^\perp}{i\lambda_n 2^{[l]} |k_n|^2} \cdot k_n, \\ k_{-nl} &:= b_{nl} k_n + \frac{\nabla^\perp b_{nl}}{-i\lambda_n 2^{[l]}} - \nabla^\perp \left(\frac{\nabla b_{nl} \cdot k_n^\perp}{\lambda_n^2 2^{2[l]} |k_n|^2} \right) + \frac{\nabla b_{nl} \cdot k_n^\perp}{-i\lambda_n 2^{[l]} |k_n|^2} \cdot k_n, \end{aligned}$$

then

$$w_{nl} = k_{nl} e^{i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + k_{-nl} e^{-i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)}. \quad (7.10)$$

Furthermore, we have

$$w_{nol}, w_{ncl}, w_{nl}, w_{no}, w_{nc}, w_n, k_{nl}, k_{-nl} \in C_c^\infty(Q_{r+\delta}).$$

Thus, we complete the construction of perturbation w_n .

7.2. Construction of n-th perturbation on temperature.

To construct χ_n , we denote β_{nl} by

$$\beta_{nl}(t, x) := \begin{cases} \frac{c_{2\ell}(t, x) \alpha_l(\mu_2 t, \mu_2 x)}{\sqrt{2e(t, x) \gamma_{k_2} \left(Id - \frac{R_{0\ell}(t, x)}{e(t, x)} \right)}}, & n = 2 \\ 0, & n = 3. \end{cases} \quad (7.11)$$

Since $\operatorname{supp} c_{2\ell} \subseteq \operatorname{supp} e$, then $\beta_{2l}(x, t)$ is well-defined.

Then we denote main l -perturbation χ_{nol} by

$$\chi_{nol}(t, x) := \begin{cases} \beta_{2l}(t, x) \left(e^{i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)} + e^{-i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)} \right), & n = 2 \\ 0, & n = 3. \end{cases}$$

and l -correction χ_{ncl} by

$$\chi_{ncl}(t, x) := \begin{cases} \Delta \beta_{2l}(t, x) \left(\frac{e^{i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)}}{-\lambda_2^2 2^{2[l]} |k_2|^2} + \frac{e^{-i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)}}{-\lambda_2^2 2^{2[l]} |k_2|^2} \right) + 2\nabla \beta_{2l}(t, x) \cdot \\ \nabla \left(\frac{e^{i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)}}{-\lambda_2^2 2^{2[l]} |k_2|^2} + \frac{e^{-i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)}}{-\lambda_2^2 2^{2[l]} |k_2|^2} \right), & n = 2 \\ 0, & n = 3. \end{cases}$$

Finally, we introduce χ_{nl} by

$$\chi_{nl} := \chi_{nol} + \chi_{ncl} = \begin{cases} \Delta \left(\beta_{2l} \left(\frac{e^{i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)}}{-\lambda_2^2 2^{2[l]} |k_2|^2} + \frac{e^{-i\lambda_2 2^{[l]} k_2^\perp \cdot (x - v_{01}(\frac{l}{\mu_2})t)}}{-\lambda_2^2 2^{2[l]} |k_2|^2} \right) \right), & n = 2 \\ 0, & n = 3. \end{cases}$$

and $\chi_{no}, \chi_{nc}, \chi_n$ by, respectively,

$$\chi_{no} := \sum_{l \in \mathbb{Z}^3} \chi_{nol}, \quad \chi_{nc} := \sum_{l \in \mathbb{Z}^3} \chi_{ncl}, \quad \chi_n := \sum_{l \in \mathbb{Z}^3} \chi_{nl}.$$

Then $\chi_{nol}, \chi_{ncl}, \chi_{nl}$ and χ_n are all real scalar functions and as the perturbations of w_n , the summation in their definitions is finite.

Now we set

$$h_{nl} := \begin{cases} \beta_{2l} - \frac{\Delta \beta_{2l}}{\lambda_2^2 2^{2[l]} |k_2|^2} + 2 \frac{\nabla \beta_{2l} \cdot k_2^\perp}{i\lambda_2 2^{[l]} |k_2|^2}, & n = 2 \\ 0, & n = 3. \end{cases} \quad h_{-nl} := \begin{cases} \beta_{2l} - \frac{\Delta \beta_{2l}}{\lambda_2^2 2^{2[l]} |k_2|^2} + 2 \frac{\nabla \beta_{2l} \cdot k_2^\perp}{-i\lambda_2 2^{[l]} |k_2|^2}, & n = 2 \\ 0, & n = 3. \end{cases}$$

then

$$\chi_{nl} = h_{nl} e^{i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + h_{-nl} e^{-i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)}.$$

Moreover, since $\text{supp } c_{2l} \subseteq Q_{r+\delta}$, we know that for all $l \in Z^3$,

$$\beta_{nl} \in C_c^\infty(Q_{r+\delta}), \quad h_{nl} \in C_c^\infty(Q_{r+\delta}) \quad (7.12)$$

and

$$\chi_{nol}, \quad \chi_{ncl}, \quad \chi_{nl}, \quad \chi_{no}, \quad \chi_{nc}, \quad \chi_n \in C_c^\infty(Q_{r+\delta}).$$

Then, by (4.5), (4.6), (4.7), (4.10) and (7.11), we know that

$$\|\beta_{nl}\|_0 \leq \begin{cases} \frac{M\sqrt{\delta}}{300}, & n = 2 \\ 0, & n = 3. \end{cases}$$

Therefore, by (4.1)

$$\|\chi_{no}\|_0 \leq \begin{cases} \frac{M\sqrt{\delta}}{4}, & n = 2 \\ 0, & n = 3. \end{cases} \quad (7.13)$$

7.3. The construction of $v_{0n}, p_{0n}, \theta_{0n}, f_{0n}, R_{0n}$.

First, we denote M_n by

$$\begin{aligned} M_n &:= \sum_{l \in Z^3} b_{nl}^2 k_n \otimes k_n \left(e^{2i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + e^{-2i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right) \\ &\quad + \sum_{l, l' \in Z^3, l \neq l'} w_{nol} \otimes w_{nol'} \end{aligned}$$

and N_n, K_n by

$$\begin{aligned} N_n &= \sum_{l \in Z^3} \left[w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) + \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \otimes w_{nl} \right], \\ K_n &= \sum_{l \in Z^3} \beta_{nl} b_{nl} k_n \left(e^{2i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + e^{-2i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right) + \sum_{l, l' \in Z^3, l \neq l'} w_{nol} \chi_{nol'}. \end{aligned}$$

Then we set

$$\begin{aligned} v_{0n} &:= v_{0(n-1)} + w_n, \quad p_{0n} := p_{0(n-1)}, \quad \theta_{0n} := \theta_{0(n-1)} + \chi_n, \\ R_{0n} &:= R_{0(n-1)} + 2 \sum_{l \in Z^3} b_{nl}^2 k_n \otimes k_n + \delta R_{0n}, \quad f_{0n} := f_{0(n-1)} + 2 \sum_{l \in Z^3} \beta_{nl} b_{nl} k_n + \delta f_{0n}, \end{aligned} \quad (7.14)$$

where

$$\begin{aligned} \delta R_{0n} &= \mathcal{R}(\text{div } M_n) + N_n - \mathcal{R}(\chi_n e_2) + \mathcal{R} \left\{ \partial_t w_n + \text{div} \left[\sum_{l \in Z^3} \left(w_{nl} \otimes v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right. \right. \right. \\ &\quad \left. \left. \left. + v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \otimes w_{nl} \right) \right] \right\} + (w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}), \end{aligned}$$

and

$$\delta f_{0n} = \begin{cases} \mathcal{G}(\text{div } K_2) + \mathcal{G} \left(\partial_t \chi_2 + \sum_{l \in Z^3} v_{01} \left(\frac{l}{\mu_2} \right) \cdot \nabla \chi_{2l} \right) + w_{2o} \chi_{2c} + \sum_{l \in Z^3} \left(v_{01} - v_{01} \left(\frac{l}{\mu_2} \right) \right) \chi_{2l} \\ \quad + w_{2c} \chi_2 + \sum_{l \in Z^3} w_{2l} \left(\theta_{01} - \theta_{01} \left(\frac{l}{\mu_2} \right) \right), & n = 2 \\ \sum_{l \in Z^3} w_{3l} \left(\theta_{02} - \theta_{02} \left(\frac{l}{\mu_3} \right) \right), & n = 3. \end{cases} \quad (7.15)$$

Since

$$\operatorname{div} M_n, \chi_n e_2, \partial_t w_n, \operatorname{div} \left[\sum_{l \in Z^3} \left(w_{nl} \otimes v_{0(n-1)} \left(\frac{l}{\mu_n} \right) + v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \otimes w_{nl} \right) \right] \in \Xi,$$

so δR_{0n} is well-defined. Moreover,

$$\operatorname{div} K_2, \partial_t \chi_2 + \sum_{l \in Z^3} v_{01} \left(\frac{l}{\mu_2} \right) \cdot \nabla \chi_{2l} \in \Psi,$$

thus δf_{0n} is well-defined. By Proposition 3.1, we know that δR_{0n} is a symmetric matrix and $\delta R_{0n} \in C_c^\infty(Q_{r+\delta})$. Also, by Proposition 3.2, we have $\delta f_{0n} \in C_c^\infty(Q_{r+\delta})$. Obviously,

$$\operatorname{div} v_{0n} = \operatorname{div} v_{0(n-1)} + \operatorname{div} w_n = 0.$$

Moreover, from the definition (7.14) on $(v_{0n}, p_{0n}, \theta_{0n}, R_{0n}, f_{0n})$ and the fact that $(v_{0(n-1)}, p_{0(n-1)}, \theta_{0(n-1)}, R_{0(n-1)}, f_{0(n-1)})$ solves the system (2.1), Proposition 3.1, we know that

$$\begin{aligned} \operatorname{div} R_{0n} &= \operatorname{div} R_{0(n-1)} + \partial_t w_n - \chi_n e_2 + \operatorname{div}(w_{no} \otimes w_{no} + w_n \otimes v_{0(n-1)} + v_{0(n-1)} \otimes w_n \\ &\quad + w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}) \\ &= \partial_t v_{0(n-1)} + \operatorname{div}(v_{0(n-1)} \otimes v_{0(n-1)}) + \nabla p_{0(n-1)} - \theta_{0(n-1)} e_2 + \partial_t w_n - \chi_n e_2 \\ &\quad + \operatorname{div}(w_{no} \otimes w_{no} + w_n \otimes v_{0(n-1)} + v_{0(n-1)} \otimes w_n + w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}) \\ &= \partial_t v_{0n} + \operatorname{div}(v_{0n} \otimes v_{0n}) + \nabla p_{0n} - \theta_{0n} e_2. \end{aligned}$$

Where we used

$$\operatorname{div}(M_n) + \operatorname{div} \left(2 \sum_{l \in Z^3} b_{nl}^2 k_n \otimes k_n \right) = \operatorname{div}(w_{no} \otimes w_{no}).$$

Furthermore, by (7.14) and (7.15), we have

$$\begin{aligned} f_{02} &= f_{01} + 2 \sum_{l \in Z^3} \beta_{2l} b_{2l} k_2 + \mathcal{G}(\operatorname{div} K_2) + w_{2o} \chi_{2c} + \mathcal{G} \left(\partial_t \chi_2 + \sum_{l \in Z^3} v_{01} \left(\frac{l}{\mu_2} \right) \cdot \nabla \chi_{2l} \right) \\ &\quad + \sum_{l \in Z^3} \left(v_0 - v_{01} \left(\frac{l}{\mu_2} \right) \right) \chi_{2l} + w_{2c} \chi_2 + \sum_{l \in Z^3} w_{2l} \left(\theta_{01} - \theta_{01} \left(\frac{l}{\mu_2} \right) \right). \end{aligned}$$

From the fact that $(v_{01}, p_{01}, \theta_{01}, R_{01}, f_{01})$ solves the system (2.1) and Proposition 3.2 we have

$$\begin{aligned} \operatorname{div} f_{02} &= \operatorname{div} f_{01} + \partial_t \chi_2 + \operatorname{div}(w_{2o} \chi_2 + w_{2c} \chi_2 + v_{01} \chi_2 + w_2 \theta_{01}) \\ &= \operatorname{div}(v_{01} \theta_{01} + w_{2o} \chi_2 + w_{2c} \chi_2 + v_{01} \chi_2 + w_2 \theta_{01}) + \partial_t (\theta_{01} + \chi_2) \\ &= \partial_t \theta_{02} + \operatorname{div}(v_{02} \theta_{02}), \end{aligned}$$

where we used

$$\operatorname{div} K_2 + \operatorname{div} \left(2 \sum_{l \in Z^3} \beta_{2l} b_{2l} k_2 \right) = \operatorname{div}(w_{2o} \chi_{2o}).$$

Thus, the functions $(v_{02}, p_{02}, \theta_{02}, R_{02}, f_{02})$ satisfies the system (2.1). And from the definition (7.14) on δf_{03} , we have

$$\operatorname{div} f_{03} = \operatorname{div} f_{02} + \operatorname{div} \delta f_{03} = \partial_t \theta_{02} + v_{02} \cdot \nabla \theta_{02} + w_3 \cdot \nabla \theta_{02} = \partial_t \theta_{03} + v_{03} \cdot \nabla \theta_{03}.$$

Thus the functions $(v_{03}, p_{03}, \theta_{03}, R_{03}, f_{03})$ also solves the system (2.1).

8. THE N-TH REPRESENTATION

In this section, we will calculate the form of

$$R_{0n} + 2 \sum_{l \in Z^3} b_{nl}^2 k_n \otimes k_n = \tilde{I}$$

and

$$f_{0n} + 2 \sum_{l \in Z^3} \beta_{nl} b_{nl} k_n = \tilde{I}I.$$

8.1. **The term \tilde{I} .**

First, by (7.5), we have

$$2 \sum_{l \in Z^3} b_{nl}^2 k_n \otimes k_n = \sum_{l \in Z^3} \alpha_l^2 (\mu_n t, \mu_n x) a_n^2 k_n \otimes k_n = a_n^2 k_n \otimes k_n.$$

Where we used $\sum_{l \in Z^3} \alpha_l^2 = 1$. Therefore, by (7.1), we have

$$R_{0(n-1)} + 2 \sum_{l \in Z^3} b_{nl}^2 k_n \otimes k_n = - \sum_{i=n+1}^3 a_i^2 k_i \otimes k_i + \sum_{i=i}^{n-1} \delta R_{0i}.$$

Meanwhile, by (7.14), we have

$$R_{0n} = - \sum_{i=n+1}^3 a_i^2(t, x) k_i \otimes k_i + \sum_{i=i}^n \delta R_{0i}.$$

In particular,

$$R_{03} = \sum_{i=1}^3 \delta R_{0i}.$$

In next section, we will prove that δR_{0n} is small.

8.2. **The term $\tilde{I}I$.**

Then, by (7.5) and (7.11), we have

$$2 \sum_{l \in Z^3} \beta_{2l} b_{2l} k_2 = \sum_{l \in Z^3} \alpha_l^2 (\mu_2 t, \mu_2 x) c_2 k_2 = c_2 k_2.$$

From the identity (5.2), we have

$$f_{01} + 2 \sum_{l \in Z^3} \beta_{2l} b_{2l} k_2 = \delta f_{01}.$$

Meanwhile, by (7.14), we have

$$f_{02} = f_{01} + 2 \sum_{l \in Z^3} \beta_{2l} b_{2l} k_2 + \delta f_{02} = \sum_{i=1}^2 \delta f_{0i}.$$

Since $\beta_{nl} = 0$ when $n = 3$, then

$$f_{03} = f_{02} + 2 \sum_{l \in Z^3} \beta_{3l} b_{3l} k_3 + \delta f_{03} = \sum_{i=1}^3 \delta f_{0i}.$$

In next section, we will prove that δg_{0n} is small.

9. ESTIMATES ON δR_{0n} AND δg_{0n}

We summarize the main estimates of b_{nl} and β_{nl} .

Lemma 9.1. *For any $l \in Z^3$ and any integer $m \geq 1$, we have*

$$\|b_{nl}\|_m + \|\beta_{nl}\|_m \leq C_m \sqrt{\delta} (\mu_n^m + \mu_n \ell^{-(m-1)}), \quad (9.1)$$

$$\|\partial_t b_{nl}\|_m + \|\partial_t \beta_{nl}\|_m \leq C_m \sqrt{\delta} (\mu_n^{m+1} + \mu_n \ell^{-m}), \quad (9.2)$$

$$\|\partial_{tt} b_{nl}\|_m + \|\partial_{tt} \beta_{nl}\|_m \leq C_m \sqrt{\delta} (\mu_n^{m+2} + \mu_n \ell^{-(m-1)}), \quad (9.3)$$

$$\|k_{\pm nl}\|_m + \|h_{\pm nl}\|_m \leq C_m \sqrt{\delta} (\mu_n^m + \mu_n \ell^{-(m-1)}), \quad (9.4)$$

$$\|\partial_t k_{\pm nl}\|_m + \|\partial_t h_{\pm nl}\|_m \leq C_m \sqrt{\delta} (\mu_n^{m+1} + \mu_n \ell^{-m}), \quad (9.5)$$

$$\|\partial_{tt} k_{\pm nl}\|_m + \|\partial_{tt} h_{\pm nl}\|_m \leq C_m \sqrt{\delta} (\mu_n^{m+2} + \mu_n \ell^{-(m-1)}) \quad (9.6)$$

and

$$\|b_{nl}\|_0 + \|\beta_{nl}\|_0 \leq C_0 \sqrt{\delta}, \quad (9.7)$$

$$\|\partial_t b_{nl}\|_0 + \|\partial_t \beta_{nl}\|_0 \leq C_0 \sqrt{\delta} \mu_n, \quad (9.8)$$

$$\|\partial_{tt} b_{nl}\|_0 + \|\partial_{tt} \beta_{nl}\|_0 \leq C_0 \sqrt{\delta} (\mu_n^2 + \mu_n \ell^{-1}), \quad (9.9)$$

$$\|k_{\pm nl}\|_0 + \|h_{\pm nl}\|_0 \leq C_0 \sqrt{\delta}, \quad (9.10)$$

$$\|\partial_t k_{\pm nl}\|_0 + \|\partial_t h_{\pm nl}\|_0 \leq C_0 \sqrt{\delta} \mu_n, \quad (9.11)$$

$$\|\partial_{tt} k_{\pm nl}\|_0 + \|\partial_{tt} h_{\pm nl}\|_0 \leq C_0 \sqrt{\delta} (\mu_n^2 + \mu_n \ell^{-1}). \quad (9.12)$$

Proof. The proof is similar to lemma 6.1, we omit the detail here. \square

Next, we give estimates on perturbations $w_{no}, w_{nc}, \chi_{no}, \chi_{nc}$.

Lemma 9.2 (Estimate on perturbation).

$$\begin{aligned} \|w_{no}\|_0 &\leq C_0 \sqrt{\delta}, \quad \|w_{no}\|_{C_{t,x}^1} \leq C_0 \sqrt{\delta} \lambda_n, \quad \|\chi_{no}\|_0 \leq C_0 \sqrt{\delta}, \quad \|\chi_{no}\|_{C_{t,x}^1} \leq C_0 \sqrt{\delta} \lambda_n, \\ \|w_{nc}\|_0 &\leq C_0 \frac{\sqrt{\delta} \mu_n}{\lambda_n}, \quad \|w_{nc}\|_{C_{t,x}^1} \leq C_0 \sqrt{\delta} \mu_n, \quad \|\chi_{nc}\|_0 \leq C_0 \frac{\sqrt{\delta} \mu_n}{\lambda_n}, \quad \|\chi_{nc}\|_{C_{t,x}^1} \leq C_0 \sqrt{\delta} \mu_n. \end{aligned} \quad (9.13)$$

Proof. The proof is similar to lemma 6.2, we omit the detail here. \square

Corollary 9.3.

$$\begin{aligned} \|v_{0n} - v_{0(n-1)}\|_0 &\leq \frac{M\sqrt{\delta}}{6} + C_0 \frac{\sqrt{\delta} \mu_n}{\lambda_n}, \quad \|v_{0n} - v_{0(n-1)}\|_{C_{t,x}^1} \leq C_0 \lambda_n \sqrt{\delta}, \quad p_{0n} - p_{0(n-1)} = 0, \\ \|\theta_{02} - \theta_{01}\|_0 &\leq \frac{M\sqrt{\delta}}{4} + C_0 \frac{\sqrt{\delta} \mu_2}{\lambda_2}, \quad \|\theta_{02} - \theta_{01}\|_{C_{t,x}^1} \leq C_0 \sqrt{\delta} \lambda_2, \quad \theta_{03} - \theta_{02} = 0. \end{aligned}$$

9.1. Estimates on δR_{0n} .

Recalling that

$$\begin{aligned} \delta R_{0n} &= \mathcal{R}(\operatorname{div} M_n) + N_n - \mathcal{R}(\chi_n e_2) + \mathcal{R}\left\{\partial_t w_n + \operatorname{div}\left[\sum_{l \in Z^3} \left(w_{nl} \otimes v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right.\right.\right. \\ &\quad \left.\left.\left.+ v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \otimes w_{nl}\right)\right]\right\} + (w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}). \end{aligned}$$

Again, we split the stress into three parts:

(1) The oscillation part

$$\mathcal{R}(\operatorname{div} M_n) - \mathcal{R}(\chi_n e_2).$$

(2) The transport part

$$\begin{aligned} & \mathcal{R}\left\{\partial_t w_n + \operatorname{div}\left[\sum_{l \in Z^3} \left(w_{nl} \otimes v_{0(n-1)}\left(\frac{l}{\mu_n}\right) + v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \otimes w_{nl}\right)\right]\right\} \\ &= \mathcal{R}\left(\partial_t w_n + \sum_{l \in Z^3} v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \cdot \nabla w_{nl}\right). \end{aligned}$$

(3) The error part

$$N_n + (w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}).$$

In the following we will estimate each term separately. Beside the estimates of N_n , the proof of other estimates are same as in section 6. We only give the proof of the estimates on N_n and omit the others here.

Lemma 9.4 (The oscillation part).

$$\begin{aligned} \|\mathcal{R}(\operatorname{div} M_n)\|_0 &\leq C_0(\varepsilon) \frac{\delta \mu_n}{\lambda_n}, \quad \|\mathcal{R}(\chi_n e_2)\|_0 \leq C_0(\varepsilon) \frac{\sqrt{\delta}}{\lambda_n}, \\ \|\mathcal{R}(\operatorname{div} M_n)\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \delta \mu_n, \quad \|\mathcal{R}(\chi_n e_2)\|_{C_{t,x}^1} \leq C_0(\varepsilon) \sqrt{\delta}. \end{aligned} \quad (9.14)$$

Lemma 9.5 (The transport part).

$$\begin{aligned} \left\| \mathcal{R}\left(\partial_t w_n + \sum_{l \in Z^3} v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \cdot \nabla w_{nl}\right) \right\|_0 &\leq C_0(\varepsilon) \frac{\sqrt{\delta} \mu_n}{\lambda_n}, \\ \left\| \mathcal{R}\left(\partial_t w_n + \sum_{l \in Z^3} v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \cdot \nabla w_{nl}\right) \right\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \sqrt{\delta} \mu_n. \end{aligned} \quad (9.15)$$

Lemma 9.6 (Estimate on error part I).

$$\|N_n\|_0 \leq C_0 \delta \frac{\lambda_{(n-1)}}{\mu_n}, \quad \|N_n\|_{C_{t,x}^1} \leq C_0 \lambda_n \delta \frac{\lambda_{(n-1)}}{\mu_n}. \quad (9.16)$$

Proof. First, we have

$$N_n = \sum_{l \in Z^3} \left[w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right) + \left(v_{0(n-1)} - v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right) \otimes w_{nl} \right].$$

By (7.10), we have

$$\begin{aligned} & \sum_{l \in Z^3} w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right) \\ &= \sum_{l \in Z^3} \left(k_{nl} e^{i\lambda_n 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} + k_{-nl} e^{-i\lambda_1 2^{[l]} k_n^\perp \cdot (x - v_{0(n-1)}(\frac{l}{\mu_n})t)} \right) \otimes \left(v_{0(n-1)} - v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right). \end{aligned}$$

Obviously, $k_{nl}(x, t) \neq 0$ implies $|(\mu_n t, \mu_n x) - l| \leq 1$. Moreover, by (7.2) and parameter assumption (4.2), we get $\|\nabla_{t,x} v_{0(n-1)}\|_0 \leq C_0 \sqrt{\delta} \lambda_{n-1}$, therefore

$$\left| k_{nl} \left(v_{0(n-1)} - v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right) \right| \leq C_0 \sqrt{\delta} \frac{\|\nabla_{t,x} v_{0(n-1)}\|_0}{\mu_n} \leq C_0 \delta \frac{\lambda_{(n-1)}}{\mu_n}.$$

Similarly,

$$\left| k_{-nl} \left(v_{0(n-1)} - v_{0(n-1)}\left(\frac{l}{\mu_n}\right) \right) \right| \leq C_0 \sqrt{\delta} \frac{\|\nabla_{t,x} v_{0(n-1)}\|_0}{\mu_n} \leq C_0 \delta \frac{\lambda_{(n-1)}}{\mu_n}.$$

Together with (4.1), it is easy to see

$$\left\| \sum_{l \in Z^3} w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \right\|_0 \leq C_0 \delta \frac{\lambda_{(n-1)}}{\mu_n}.$$

Applying the same argument

$$\left\| \sum_{l \in Z^3} \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \otimes w_{nl} \right\|_0 \leq C_0 \delta \frac{\lambda_{(n-1)}}{\mu_n}.$$

Thus, we arrive at the first estimate in this lemma. A straightforward computation gives

$$\begin{aligned} & \partial_t \left(\sum_{l \in Z^3} w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \right) \\ &= \sum_{l \in Z^3} \partial_t w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) + \sum_{l \in Z^3} w_{nl} \otimes \partial_t v_{0(n-1)}. \end{aligned}$$

Thus, by (7.2), parameter assumption (4.2) and Corollary 9.3

$$\left\| \partial_t \left(\sum_{l \in Z^3} w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \right) \right\|_0 \leq C_0 \lambda_n \delta \frac{\lambda_{(n-1)}}{\mu_n}.$$

The same argument gives

$$\begin{aligned} & \left\| \nabla \left(\sum_{l \in Z^3} w_{nl} \otimes \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \right) \right\|_0 \leq C_0 \lambda_n \delta \frac{\lambda_{(n-1)}}{\mu_n}, \\ & \left\| \partial_t \left(\sum_{l \in Z^3} \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \otimes w_{nl} \right) \right\|_0 \leq C_0 \lambda_n \delta \frac{\lambda_{(n-1)}}{\mu_n}, \\ & \left\| \nabla \left(\sum_{l \in Z^3} \left(v_{0(n-1)} - v_{0(n-1)} \left(\frac{l}{\mu_n} \right) \right) \otimes w_{nl} \right) \right\|_0 \leq C_0 \lambda_n \delta \frac{\lambda_{(n-1)}}{\mu_n}. \end{aligned}$$

Finally, collecting these estimates, we arrive at

$$\|N_n\|_{C_{t,x}^1} \leq C_0 \lambda_n \delta \frac{\lambda_{(n-1)}}{\mu_n}.$$

Thus, the proof of this lemma is complete. \square

Lemma 9.7 (Estimates on error part II).

$$\begin{aligned} & \|w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}\|_0 \leq C_0 \frac{\delta \mu_n}{\lambda_n}, \\ & \|w_{no} \otimes w_{nc} + w_{nc} \otimes w_{no} + w_{nc} \otimes w_{nc}\|_{C_{t,x}^1} \leq C_0 \delta \mu_n. \end{aligned}$$

From Lemma 9.4, Lemma 9.5, Lemma 9.6 and Lemma 9.7, we conclude that

$$\|\delta R_{0n}\|_0 \leq C_0(\varepsilon) \left(\frac{\sqrt{\delta} \mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|\delta R_{0n}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_n \left(\frac{\sqrt{\delta} \mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right).$$

9.2. Estimate on δf_{0n} .

We first deal with δf_{02} . Recalling that

$$\begin{aligned} \delta f_{02} &= \mathcal{G}(\operatorname{div} K_2) + \mathcal{G} \left(\partial_t \chi_2 + \sum_{l \in Z^3} v_{01} \left(\frac{l}{\mu_2} \right) \cdot \nabla \chi_{2l} \right) + w_{2o} \chi_{2c} + \sum_{l \in Z^3} \left(v_{01} - v_{01} \left(\frac{l}{\mu_2} \right) \right) \chi_{2l} \\ &\quad + w_{2c} \chi_2 + \sum_{l \in Z^3} w_{2l} \left(\theta_{01} - \theta_{01} \left(\frac{l}{\mu_2} \right) \right). \end{aligned}$$

As before, we split δf_{02} into three parts:

(1) The oscillation part:

$$\mathcal{G}(\operatorname{div} K_2).$$

(2) The transport part:

$$\mathcal{G}\left(\partial_t \chi_2 + \sum_{l \in Z^3} v_{01}\left(\frac{l}{\mu_2}\right) \cdot \nabla \chi_{2l}\right).$$

(3) The error part:

$$w_{2c} \chi_2 + w_{2o} \chi_{2c} + \sum_{l \in Z^3} \left(v_{01} - v_{01}\left(\frac{l}{\mu_2}\right)\right) \chi_{2l} + \sum_{l \in Z^3} w_{2l} \left(\theta_{01} - \theta_{01}\left(\frac{l}{\mu_2}\right)\right).$$

Lemma 9.8 (The oscillation part).

$$\|\mathcal{G}(\operatorname{div} K_2)\|_0 \leq C_0(\varepsilon) \frac{\delta \mu_2}{\lambda_2}, \quad \|\mathcal{G}(\operatorname{div} K_2)\|_{C_{t,x}^1} \leq C_0(\varepsilon) \delta \mu_2.$$

Lemma 9.9 (The transport part).

$$\begin{aligned} \left\| \mathcal{G}\left(\partial_t \chi_2 + \sum_{l \in Z^3} v_{01}\left(\frac{l}{\mu_2}\right) \cdot \nabla \chi_{2l}\right) \right\|_0 &\leq C_0(\varepsilon) \frac{\sqrt{\delta} \mu_2}{\lambda_2}, \\ \left\| \mathcal{G}\left(\partial_t \chi_2 + \sum_{l \in Z^3} v_{01}\left(\frac{l}{\mu_2}\right) \cdot \nabla \chi_{2l}\right) \right\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \sqrt{\delta} \mu_2. \end{aligned}$$

Their proofs are same as in section 6.

Lemma 9.10 (The error part).

$$\begin{aligned} \left\| w_{2c} \chi_2 + w_{2o} \chi_{2c} + \sum_{l \in Z^3} \left(\theta_{01} - \theta_{01}\left(\frac{l}{\mu_2}\right)\right) \chi_{2l} \right\|_0 &\leq C_0 \delta \left(\frac{\mu_2}{\lambda_2} + \frac{\lambda_1}{\mu_2}\right), \\ \left\| w_{2c} \chi_2 + w_{2o} \chi_{2c} + \sum_{l \in Z^3} \left(\theta_{01} - \theta_{01}\left(\frac{l}{\mu_2}\right)\right) \chi_{2l} \right\|_{C_{t,x}^1} &\leq C_0 \lambda_2 \delta \left(\frac{\mu_2}{\lambda_2} + \frac{\lambda_1}{\mu_2}\right). \end{aligned}$$

Proof. First, by Lemma 9.2

$$\|w_{2c} \chi_2 + w_{2o} \chi_{2c}\|_0 \leq C_0 \delta \frac{\mu_2}{\lambda_2}.$$

As the argument in Lemma 9.6, we have

$$\left\| \sum_{l \in Z^3} \left(v_{01} - v_{01}\left(\frac{l}{\mu_2}\right)\right) \chi_{2l} \right\|_0 \leq C_0 \delta \frac{\lambda_1}{\mu_2}, \quad \left\| \sum_{l \in Z^3} w_{2l} \left(\theta_{01} - \theta_{01}\left(\frac{l}{\mu_2}\right)\right) \right\|_0 \leq C_0 \delta \frac{\lambda_1}{\mu_2}.$$

The $C_{t,x}^1$ estimate is similar to that of Lemma 9.6. □

From the above three lemmas, we conclude

$$\|\delta f_{02}\|_0 \leq C_0(\varepsilon) \left(\frac{\sqrt{\delta} \mu_2}{\lambda_2} + \delta \frac{\lambda_1}{\mu_2}\right), \quad \|\delta f_{02}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_2 \left(\frac{\sqrt{\delta} \mu_2}{\lambda_2} + \delta \frac{\lambda_1}{\mu_2}\right).$$

Now we consider the estimates of δf_{03} . From definition (7.15) on δf_{03} , applying the same argument as in Lemma 6.10, we have

$$\|\delta f_{0n}\|_0 \leq C_0 \delta \frac{\lambda_2}{\mu_3}, \quad \|\delta f_{0n}\|_{C_{t,x}^1} \leq C_0 \lambda_3 \delta \frac{\lambda_2}{\mu_3}.$$

By inductions, we know that, for any $2 \leq n \leq 3$, $(v_{0n}, p_{0n}, \theta_{0n}, R_{0n}, f_{0n}) \in C_c^\infty(Q_{r+\delta})$ solves system (2.1) and satisfies

$$R_{0n} = - \sum_{i=n+1}^3 a_i^2 k_i \otimes k_i + \sum_{i=i}^n \delta R_{0i}, \quad g_{0n} := \sum_{i=1}^n \delta g_{0i}$$

with the estimates

$$\begin{aligned} \|v_{0n} - v_{0(n-1)}\|_0 &\leq \frac{M\sqrt{\delta}}{6} + C_0 \frac{\sqrt{\delta}\mu_n}{\lambda_n}, \quad \|v_{0n} - v_{0(n-1)}\|_{C_{t,x}^1} \leq C_0 \lambda_n \sqrt{\delta}, \quad \|p_{0n} - p_{0(n-1)}\|_0 = 0, \\ \|\theta_{0n} - \theta_{0(n-1)}\|_0 &\leq \begin{cases} \frac{M\sqrt{\delta}}{4} + C_0 \frac{\sqrt{\delta}\mu_2}{\lambda_2}, & n = 2 \\ 0, & n = 3. \end{cases} \quad \|\theta_{0n} - \theta_{0(n-1)}\|_{C_{t,x}^1} \leq \begin{cases} C_0 \lambda_2 \sqrt{\delta}, & n = 2 \\ 0, & n = 3. \end{cases} \\ \|\delta R_{0n}\|_0 &\leq C_0(\varepsilon) \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|\delta f_{0n}\|_0 \leq C_0(\varepsilon) \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \\ \|\delta R_{0n}\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \lambda_n \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|\delta f_{0n}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \lambda_n \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right). \end{aligned}$$

Finally, we obtain $(v_{03}, p_{03}, \theta_{03}, R_{03}, f_{03}) \in C_c^\infty(Q_{r+\delta})$ which solves system (2.1) and satisfies

$$\begin{aligned} \|R_{03}\|_0 &\leq C_0(\varepsilon) \sum_{n=1}^3 \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|f_{03}\|_0 \leq C_0(\varepsilon) \sum_{n=1}^3 \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \\ \|R_{03}\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \sum_{n=1}^3 \lambda_n \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|f_{03}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \sum_{n=1}^3 \lambda_n \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \\ \|v_{03} - v_0\|_0 &\leq \frac{M\sqrt{\delta}}{2} + C_0 \sum_{n=1}^3 \frac{\sqrt{\delta}\mu_n}{\lambda_n}, \quad \|v_{03} - v_0\|_{C_{t,x}^1} \leq C_0 \sum_{n=1}^3 \lambda_n \sqrt{\delta}, \\ \|p_{03} - p_0\|_0 &\leq M\delta, \quad \|p_{03} - p_0\|_{C_{t,x}^1} \leq C_0, \\ \|\theta_{03} - \theta_0\|_0 &\leq \frac{M\sqrt{\delta}}{2} + C_0 \sum_{n=1}^2 \frac{\sqrt{\delta}\mu_n}{\lambda_n}, \quad \|\theta_{03} - \theta_0\|_{C_{t,x}^1} \leq C_0 \sum_{n=1}^2 \lambda_n \sqrt{\delta}. \end{aligned}$$

10. PROOF OF PROPOSITION 2.1

In this section, we prove Proposition 2.1 by choosing the appropriate parameters ℓ, μ_n, λ_n for $1 \leq n \leq 3$.

Proof. From the results of section 9, we have $(v_{03}, p_{03}, \theta_{03}, R_{03}, f_{03}) \in C_c^\infty(Q_{r+\delta})$ which solves system (2.1) and satisfies

$$\begin{aligned} \|R_{03}\|_0 &\leq C_0(\varepsilon) \sum_{n=1}^3 \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|f_{03}\|_0 \leq C_0(\varepsilon) \sum_{n=1}^3 \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \\ \|R_{03}\|_{C_{t,x}^1} &\leq C_0(\varepsilon) \sum_{n=1}^3 \lambda_n \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \quad \|f_{03}\|_{C_{t,x}^1} \leq C_0(\varepsilon) \sum_{n=1}^3 \lambda_n \left(\frac{\sqrt{\delta}\mu_n}{\lambda_n} + \delta \frac{\lambda_{n-1}}{\mu_n} \right), \end{aligned}$$

$$\begin{aligned}
\|v_{03} - v_0\|_0 &\leq \frac{M\sqrt{\delta}}{2} + C_0 \sum_{n=1}^3 \frac{\sqrt{\delta}\mu_n}{\lambda_n}, & \|v_{03} - v_0\|_{C_{t,x}^1} &\leq C_0 \sum_{n=1}^3 \lambda_n \sqrt{\delta}, \\
\|p_{03} - p_0\|_0 &\leq M\delta, & \|p_{03} - p_0\|_{C_{t,x}^1} &\leq C_0, \\
\|\theta_{03} - \theta_0\|_0 &\leq \frac{M\sqrt{\delta}}{2} + C_0 \sum_{n=1}^2 \frac{\sqrt{\delta}\mu_n}{\lambda_n}, & \|\theta_{03} - \theta_0\|_{C_{t,x}^1} &\leq C_0 \sum_{n=1}^2 \lambda_n \sqrt{\delta}.
\end{aligned} \tag{10.1}$$

where $\lambda_0 = \Lambda\delta^{-\frac{1}{2}} + \frac{\mu_1\Lambda\ell}{\delta}$. We divide the remainder proof into four steps:

Step 1. We now specify the choice of the parameters. First choose:

$$\ell = \frac{1}{L_v} \frac{\bar{\delta}}{\Lambda}, \tag{10.2}$$

with L_v being a sufficiently large constant, which depends only on $\|v\|_0$.

Next, we impose

$$\mu_1 = L_v \frac{\sqrt{\delta}}{\delta} \Lambda, \quad \lambda_1 = L_v \frac{\sqrt{\delta}}{\delta} \mu_1^{1+\varepsilon}, \quad \mu_i = L_v \frac{\delta\lambda_{i-1}}{\delta}, \quad \lambda_i = L_v \frac{\sqrt{\delta}}{\delta} \mu_i^{1+\varepsilon}, \quad i = 2, 3. \tag{10.3}$$

Step 2. Compatibility condition. We check that all the conditions in (4.2), (7.4) are satisfied by our choice of the parameters.

We first check the triple (ℓ, μ_1, λ_1) . By (10.2)

$$\ell^{-1} = L_v \frac{\Lambda}{\delta} \geq \frac{\Lambda}{\eta\delta}$$

if we take $L_v \geq \frac{1}{\eta}$.

Since $\bar{\delta} \leq \delta^{\frac{3}{2}}$,

$$\mu_1 \geq \frac{\Lambda}{\delta}$$

and

$$\lambda_1 \geq L_v^{1+\varepsilon} \Lambda^{1+\varepsilon} \left(\frac{\sqrt{\delta}}{\delta}\right)^{2+\varepsilon} \geq \left(L_v \frac{\Lambda}{\delta}\right)^{1+\varepsilon} \geq \ell^{-(1+\varepsilon)}.$$

It's obvious that

$$\lambda_1 \geq \mu_1^{1+\varepsilon}.$$

Thus, (4.2) is satisfied.

Next, for $i = 2, 3$, it's obvious that

$$\frac{\mu_i}{\mu_{i-1}} = \frac{\lambda_{i-1}}{\lambda_{i-2}} > 1.$$

A straightforward computation yield

$$\lambda_i \geq \frac{\sqrt{\delta}}{\delta} \left(\frac{\delta}{\delta}\right)^{1+\varepsilon} \lambda_{i-1}^{1+\varepsilon} \geq \lambda_{i-1} \geq \ell^{-(1+\varepsilon)}.$$

It's obvious that

$$\lambda_i \geq \mu_i^{1+\varepsilon}.$$

Thus, the relationship (7.4) is satisfied.

Step 3. C^0 estimates Fixed ε small. Thus, (10.1) implies

$$\begin{aligned}
\|R_{03}\|_0 &\leq C_0(\varepsilon)\bar{\delta}L_v^{-1}, \quad \|f_{03}\|_0 \leq C_0(\varepsilon)\bar{\delta}L_v^{-1}, \\
\|v_{03} - v_0\|_0 &\leq \frac{M\sqrt{\delta}}{2} + C_0\bar{\delta}L_v^{-1}, \quad \|\theta_{03} - \theta_0\|_0 \leq \frac{M\sqrt{\delta}}{2} + C_0\bar{\delta}L_v^{-1}, \quad \|p_{03} - p_0\|_0 \leq M\delta.
\end{aligned}$$

Choosing L_v sufficiently large, we can achieve the desired inequalities (2.4)-(2.8).

Step 4. C^1 estimates. By the specified choices of parameters we have

$$\|R_{03}\|_{C^1} \leq \lambda_3 \bar{\delta}, \quad \|f_{03}\|_{C^1} \leq \lambda_3 \bar{\delta}, \quad \|v_{03} - v_0\|_{C_{t,x}^1} \leq C_0 \lambda_3 \sqrt{\delta}, \quad \|\theta_{03} - \theta_0\|_{C_{t,x}^1} \leq C_0 \lambda_3 \sqrt{\delta}.$$

Notice that for $i = 2, 3$, there holds

$$\lambda_i = L_v^{2+\varepsilon} \frac{\sqrt{\delta}}{\bar{\delta}} \left(\frac{\delta}{\bar{\delta}}\right)^{1+\varepsilon} \lambda_{i-1}^{1+\varepsilon} = \frac{1}{\sqrt{\delta}} \left(\frac{D_v \delta}{\bar{\delta}}\right)^{2+\varepsilon} \lambda_{i-1}^{1+\varepsilon}.$$

Thus, we conclude

$$\begin{aligned} \max\{1, \|R_{03}\|_{C^1}, \|f_{03}\|_{C^1}, \|v_{03}\|_{C^1}, \|\theta_{03}\|_{C^1}\} &\leq \Lambda + C_0(\varepsilon) \sqrt{\delta} \lambda_3 \\ &\leq C_0(\varepsilon) L_v^{(1+\varepsilon)^2(2+\varepsilon)+(2+\varepsilon)^2} (\sqrt{\delta})^{\varepsilon^2+3\varepsilon+3} \left(\frac{\sqrt{\delta}}{\bar{\delta}}\right)^{(1+\varepsilon)^2(2+\varepsilon)+(2+\varepsilon)^2} \Lambda^{(1+\varepsilon)^3}. \end{aligned}$$

Setting $A = C_0(\varepsilon) L_v^{(1+\varepsilon)^2(2+\varepsilon)+(2+\varepsilon)^2}$, we conclude estimate (2.9).

More precisely, we have

$$\begin{aligned} \|\theta_{03}\|_{C_{t,x}^1} &\leq \Lambda + C_0(\varepsilon) \sqrt{\delta} \lambda_2 \leq C_0(\varepsilon) L_v^{4+4\varepsilon+\varepsilon^2} (\sqrt{\delta})^{2+\varepsilon} \left(\frac{\sqrt{\delta}}{\bar{\delta}}\right)^{4+4\varepsilon+\varepsilon^2} \Lambda^{(1+\varepsilon)^2} \\ &\leq A \delta^{\frac{2+\varepsilon}{2}} \left(\frac{\sqrt{\delta}}{\bar{\delta}}\right)^{4+4\varepsilon+\varepsilon^2} \Lambda^{(1+\varepsilon)^2}, \end{aligned}$$

this is the second estimate in (2.10).

Finally, we set

$$\tilde{V} := v_{03}, \quad \tilde{p} := p_{03}, \quad \tilde{\theta} := \theta_{03}, \quad \tilde{R} := R_{03}, \quad \tilde{f} := f_{03},$$

then \tilde{V} , \tilde{p} , $\tilde{\theta}$, \tilde{R} , \tilde{f} are what we need in our Proposition (2.1). \square

11. PROOF OF THEOREM 1.1

We first construct a non-trivial solution with compact support both in space and time for system (2.1).

11.1. Construction of compactly supported solution $(v_0, p_0, \theta_0, R_0, f_0)$ for system (2.1).

We first set $k_1 := (1, 0)^T$ and let

$$0 \leq \varphi(t, x) \in C_c^\infty(Q_r; R), \quad \varphi(t, x) = 10M \quad \text{in} \quad Q_{\frac{r}{2}},$$

where r, M is the constant appeared in Proposition (2.1). Then set

$$\bar{p}(t, x) := -2\varphi^2(t, x), \quad a_{1l}(t, x) := \varphi(t, x) \alpha_l(\mu_1 t, \mu_1 x), \quad \bar{R}(t, x) := \begin{pmatrix} -2\varphi^2(t, x) & 0 \\ 0 & -2\varphi^2(t, x) \end{pmatrix}.$$

Here α_l is the partition of unity in section 4. Obviously, $\nabla \bar{p} = \text{div} \bar{R}$.

We first set

$$\begin{aligned} v_{01ol}(t, x) &:= -a_{1l}(t, x) k_1 \left(i e^{i\lambda_1 2^{|l|} k_1^\perp \cdot x} - i e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x} \right), \\ v_{01cl}(t, x) &:= \nabla^\perp (\nabla a_{1l}(t, x) \cdot k_1^\perp) \left(\frac{i e^{i\lambda_1 2^{|l|} k_1^\perp \cdot x} - i e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x}}{\lambda_1^2 2^{2|l|}} \right) - \nabla^\perp a_{1l}(t, x) \left(\frac{e^{i\lambda_1 2^{|l|} k_1^\perp \cdot x} + e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x}}{\lambda_1 2^{|l|}} \right) \\ &\quad - \nabla a_{1l}(t, x) \cdot k_1^\perp \left(\frac{k_1 e^{i\lambda_1 2^{|l|} k_1^\perp \cdot x} + k_1 e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x}}{\lambda_1 2^{|l|}} \right), \end{aligned} \tag{11.1}$$

where $|l|$ is the length of l , $\mu_1 \ll \lambda_1$ are two positive numbers which will be chosen quite large, depending on appropriate norms of φ .

Then set

$$v_{01l} := v_{01ol} + v_{01cl}, \quad v_{01o} := \sum_{l \in Z^3} v_{01ol}, \quad v_{01c} := \sum_{l \in Z^3} v_{01cl}, \quad v_{01} := \sum_{l \in Z^3} v_{01l}.$$

Thus, a straightforward computation gives

$$v_{01}(t, x) := \sum_{l \in Z^3} \nabla^\perp \operatorname{div} \left(a_{1l}(t, x) \left(\frac{ik_1^\perp e^{i\lambda_1 2^{|l|} k_1^\perp \cdot x} - ik_1^\perp e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x}}{\lambda_1^2 2^{2|l|}} \right) \right).$$

Let $b(t, x) \in C_c^\infty(Q_r; R)$ and set

$$\begin{aligned} \theta_{01o}(t, x) &:= -b(t, x)(e^{i\lambda_1 k_1^\perp \cdot x} + e^{-i\lambda_1 k_1^\perp \cdot x}), \\ \theta_{01c}(t, x) &:= \Delta b(t, x) \frac{e^{i\lambda_1 k_1^\perp \cdot x} + e^{-i\lambda_1 k_1^\perp \cdot x}}{\lambda_1^2} + 2\nabla b(t, x) \cdot k_1^\perp \frac{ie^{i\lambda_1 k_1^\perp \cdot x} - ie^{-i\lambda_1 k_1^\perp \cdot x}}{\lambda_1}. \end{aligned}$$

Then denote θ_{01} by

$$\theta_{01} := \theta_{01c} + \theta_{01o}.$$

Thus

$$\theta_{01}(t, x) = \Delta \left(b(t, x) \left(\frac{e^{i\lambda_1 k_1^\perp \cdot x} + e^{-i\lambda_1 k_1^\perp \cdot x}}{\lambda_1^2} \right) \right).$$

Finally, we set

$$\begin{aligned} p_{01} &:= \bar{p}, \quad R_{01} := \bar{R} + 2 \sum_{l \in Z^3} a_{1l}^2 k_1 \otimes k_1 + \delta R_{01}, \\ f_{01} &:= v_{01o} \theta_{01c} + v_{01c} \theta_{01o} + v_{01c} \theta_{01c} + \mathcal{G} \left(\partial_t \theta_{01} + \operatorname{div}(v_{01o} \theta_{01o}) \right), \end{aligned}$$

where

$$\begin{aligned} \delta R_{01} &= v_{01o} \otimes v_{01c} + v_{01c} \otimes v_{01o} + v_{01c} \otimes v_{01c} + \mathcal{R} \left(\partial_t v_{01} \right. \\ &\quad \left. + \operatorname{div} \left(- \sum_{l \in Z^3} a_{1l}^2 k_1 \otimes k_1 \left(e^{2i\lambda_1 2^{|l|} k_1^\perp \cdot x} + e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x} \right) + \sum_{l, l' \in Z^3, l \neq l'} v_{01ol} \otimes v_{01ol'} \right) - \theta_{01} e_2 \right). \end{aligned}$$

Obviously, $\operatorname{div} v_{01} = 0$ and $\partial_t v_{01}, \theta_{01} e_2 \in \Xi$, $\partial_t \theta_{01} + \operatorname{div}(v_{01o} \theta_{01o}) \in \Psi$, thus R_{01}, f_{01} is well-defined. By Proposition 3.1 and Proposition 3.2, we know that $(v_{01}, p_{01}, \theta_{01}, R_{01}, f_{01}) \in C_c^\infty(Q_r)$ solves Boussinesq-stress system (2.1). In fact, by Proposition 3.1, we have

$$\operatorname{div} R_{01} = \partial_t v_{01} + \operatorname{div}(v_{01} \otimes v_{01}) + \nabla p_{01} - \theta_{01} e_2,$$

where we use the identity

$$\begin{aligned} &\operatorname{div} \left\{ - \sum_{l \in Z^3} a_{1l}^2 k_1 \otimes k_1 \left(e^{2i\lambda_1 2^{|l|} k_1^\perp \cdot x} + e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x} \right) + 2 \sum_{l \in Z^3} a_{1l}^2 k_1 \otimes k_1 + \sum_{l, l' \in Z^3, l \neq l'} v_{01ol} \otimes v_{01ol'} \right\} \\ &= \operatorname{div}(v_{01o} \otimes v_{01o}), \quad \operatorname{div} \bar{R} = \nabla p_{01}. \end{aligned}$$

Using Proposition 3.2, we have

$$\operatorname{div} f_{01} = \partial_t \theta_{01} + \operatorname{div}(v_{01} \theta_{01}).$$

Thus, $(v_{01}, p_{01}, \theta_{01}, R_{01}, f_{01})$ solves Boussinesq-stress system (2.1). Furthermore, we have

$$\bar{R} + 2 \sum_{l \in Z^3} a_{1l}^2 k_1 \otimes k_1 = \begin{pmatrix} -2\varphi^2 & 0 \\ 0 & -2\varphi^2 \end{pmatrix} + \begin{pmatrix} 2\varphi^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2\varphi^2 \end{pmatrix},$$

therefore

$$R_{01} = \begin{pmatrix} 0 & 0 \\ 0 & -2\varphi^2 \end{pmatrix} + \delta R_{01}. \quad (11.2)$$

We claim δR_{01} , g_{01} can be arbitrarily small by choosing appropriate μ_1 and λ_1 . In fact,

$$\|v_{01c}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}, \quad \|\theta_{01c}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}, \quad \|v_{01o}\|_0 \leq C_1, \quad \|\theta_{01o}\|_0 \leq C_1.$$

Here and subsequest, C_1 is an absolute constant which depends on functions b , φ . Therefore

$$\|v_{01o} \otimes v_{01c} + v_{01c} \otimes v_{01o} + v_{01c} \otimes v_{01c}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}, \quad \|v_{01o}\theta_{01c} + v_{01c}\theta_{01o} + v_{01c}\theta_{01c}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}. \quad (11.3)$$

Moreover, by Proposition 3.1 and the same argument as 6.4, we have

$$\begin{aligned} & \left\| \mathcal{R} \left(\partial_t v_{01} + \operatorname{div} \left(- \sum_{l \in Z^3} a_{1l}^2 k_1 \otimes k_1 \left(e^{2i\lambda_1 2^{|l|} k_1^\perp \cdot x} + e^{-i\lambda_1 2^{|l|} k_1^\perp \cdot x} \right) + \sum_{l, l' \in Z^3, l \neq l'} v_{01ol} \otimes v_{01ol'} \right) - \theta_{01} e_2 \right) \right\|_0 \\ & \leq C_1 \frac{\mu_1}{\lambda_1}. \end{aligned} \quad (11.4)$$

Similarly, by Proposition 3.2, we obtain

$$\left\| \mathcal{G} \left(\partial_t \theta_{01} + \operatorname{div}(v_{01o} \theta_{01o}) \right) \right\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}. \quad (11.5)$$

Thus, combining (11.3), (11.4) and (11.5), we arrive at

$$\|f_{01}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}, \quad \|\delta R_{01}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1}.$$

Hence δR_{01} , g_{01} can be arbitrarily small by choosing appropriate μ_1 , λ_1 .

Take $k_2 := (0, 1)^T$, $a_{2l}(t, x) := \varphi(t, x) \alpha_l(\mu_2 t, \mu_2 x)$ and set

$$\begin{aligned} w_{2ol}(t, x) &:= -a_{2l}(t, x) k_2 \left(i e^{i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)} - i e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)} \right), \\ w_{2cl}(t, x) &:= \nabla^\perp (\nabla a_{2l}(t, x) \cdot k_2^\perp) \left(\frac{i e^{i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)} - i e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)}}{\lambda_2^2 2^{2|l|}} \right) \\ &\quad - \nabla^\perp a_{2l}(t, x) \left(\frac{e^{i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)} + e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)}}{\lambda_2 2^{|l|}} \right) \\ &\quad - \nabla a_{2l}(t, x) \cdot k_2^\perp \left(\frac{k_2 e^{i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)} + k_2 e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)}}{\lambda_2 2^{|l|}} \right), \\ w_{2l} &:= w_{2ol} + w_{2cl}, \quad w_{2o} := \sum_{l \in Z^3} w_{2ol}, \quad w_{2c} := \sum_{l \in Z^3} w_{2cl}, \quad w_2 := w_{2o} + w_{2c}, \end{aligned} \quad (11.6)$$

where $\mu_2 \ll \lambda_2$ are two positive numbers which will be chosen quite large, depending on appropriate norms of v_{01}, θ_{01} . Then, a straightforward computation gives

$$w_2 := \sum_{l \in Z^3} \nabla^\perp \operatorname{div} \left(a_{2l} \left(\frac{i k_2^\perp e^{i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)} - i k_2^\perp e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot (x - v_{01}(\frac{t}{\mu_2})t)}}{\lambda_2^2 2^{2|l|}} \right) \right).$$

Finally, we set

$$\begin{aligned} v_{02} &:= v_{01} + w_2, \quad \theta_{02} := \theta_{01}, \quad p_{02} := p_{01}. \\ R_{02} &:= R_{01} + 2 \sum_{l \in Z^3} a_{2l}^2 k_2 \otimes k_2 + \delta R_{02}, \quad f_{02} := f_{01} + \delta f_{02}, \end{aligned}$$

where

$$\begin{aligned}
\delta R_{02} = & w_{2o} \otimes w_{2c} + w_{2c} \otimes w_{2o} + w_{2c} \otimes w_{2c} + w_{2c} \otimes v_{01} + v_{01} \otimes w_{2c} + \mathcal{R} \left(\partial_t w_2 + \operatorname{div} \left(w_{2o} \otimes v_{01} \left(\frac{l}{\mu_2} \right) \right. \right. \\
& \left. \left. + v_{01} \left(\frac{l}{\mu_2} \right) \otimes w_{2o} \right) \right) + \operatorname{div} \left(- \sum_{l \in Z^3} a_{2l}^2 k_2 \otimes k_2 \left(e^{2i\lambda_2 2^{|l|} k_2^\perp \cdot x} + e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot x} \right) \right. \\
& \left. + \sum_{l, l' \in Z^3, l \neq l'} w_{2ol} \otimes w_{2ol'} \right) + w_{2o} \otimes \left(v_{01} - v_{01} \left(\frac{l}{\mu_2} \right) \right) + \left(v_{01} - v_{01} \left(\frac{l}{\mu_2} \right) \right) \otimes w_{2o}, \\
\delta f_{02} = & \mathcal{G}(w_2 \cdot \nabla \theta_{01}).
\end{aligned} \tag{11.7}$$

Obviously, $\operatorname{div} v_{02} = 0$ and $\partial_t w_2 \in \Xi$, thus by Proposition 3.1 and Proposition 3.2, R_{02}, f_{02} are well-defined and $(v_{02}, p_{02}, \theta_{02}, R_{02}, f_{02}) \in C_c^\infty(Q_r)$ solves Boussinesq-stress system (2.1). In fact,

$$\begin{aligned}
\operatorname{div} R_{02} = & \operatorname{div} R_{01} + \operatorname{div}(w_{2o} \otimes w_{2o} + w_{2o} \otimes w_{2c} + w_{2c} \otimes w_{2o} + w_{2c} \otimes w_{2c} + w_{2o} \otimes v_{01} \\
& + v_{01} \otimes w_{2o} + w_{2c} \otimes v_{01} + v_{01} \otimes w_{2c}) + \partial_t w_2 \\
= & \partial_t v_{02} + \operatorname{div}(v_{02} \otimes v_{02}) + \nabla p_{02} - \theta_{02} e_2,
\end{aligned}$$

where we use

$$\begin{aligned}
& \operatorname{div} \left\{ - \sum_{l \in Z^3} a_{2l}^2 k_2 \otimes k_2 \left(e^{2i\lambda_2 2^{|l|} k_2^\perp \cdot x} + e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot x} \right) + 2 \sum_{l \in Z^3} a_{2l}^2 k_2 \otimes k_2 + \sum_{l, l' \in Z^3, l \neq l'} w_{2ol} \otimes w_{2ol'} \right\} \\
= & \operatorname{div}(w_{2o} \otimes w_{2o}).
\end{aligned}$$

And

$$\operatorname{div} f_{02} = \operatorname{div} f_{01} + \operatorname{div}(w_2 \theta_{01}) = \partial_t \theta_{01} + \operatorname{div}(v_{01} \theta_{01} + w_2 \theta_{01}) = \partial_t \theta_{02} + \operatorname{div}(v_{02} \theta_{02}).$$

We claim $\delta R_{02}, \delta g_{02}$ can be arbitrarily small by choosing appropriate μ_2 , and λ_2 . In fact,

$$\|w_{2c}\|_0 \leq C_2 \frac{\mu_2}{\lambda_2}, \quad \|w_{2o}\|_0 \leq C_2.$$

Here and subsequent, C_2 is a constant which depends on appropriate norms of v_{01}, θ_{01} . Therefore

$$\|w_{2o} \otimes w_{2c} + w_{2c} \otimes w_{2o} + w_{2c} \otimes w_{2c} + w_{2c} \otimes v_{01} + v_{01} \otimes w_{2c}\|_0 \leq C_2 \frac{\mu_2}{\lambda_2}. \tag{11.8}$$

By the same argument as lemma 6.4, Lemma 6.5, we obtain

$$\begin{aligned}
& \left\| \mathcal{R} \left\{ \operatorname{div} \left(- \sum_{l \in Z^3} a_{2l}^2 k_2 \otimes k_2 \left(e^{2i\lambda_2 2^{|l|} k_2^\perp \cdot x} + e^{-i\lambda_2 2^{|l|} k_2^\perp \cdot x} \right) + \sum_{l, l' \in Z^3, l \neq l'} w_{2ol} \otimes w_{2ol'} \right) \right\} \right\|_0 \leq C_2 \frac{\mu_2}{\lambda_2}, \\
& \left\| \mathcal{R} \left(\partial_t w_{2o} + v_{01} \left(\frac{l}{\mu_1} \right) \cdot \nabla w_{2o} \right) \right\|_0 \leq C_2 \frac{\mu_2}{\lambda_2}, \quad \left\| \mathcal{R}(\partial_t w_{2c}) \right\|_0 \leq C_2 \frac{\mu_2}{\lambda_2}.
\end{aligned}$$

Moreover, $a_{2l}(t, x) \neq 0$ implies $|(\mu_2 t, \mu_2 x) - l| \leq 1$, therefore

$$\left\| w_{2o} \otimes \left(v_{01} - v_{01} \left(\frac{l}{\mu_2} \right) \right) + \left(v_{01} - v_{01} \left(\frac{l}{\mu_2} \right) \right) \otimes w_{2o} \right\|_0 \leq \frac{C_2}{\mu_2}.$$

Collecting the above estimates, we arrive at

$$\|\delta R_{02}\|_0 \leq \frac{C_2}{\mu_2} + C_2 \frac{\mu_2}{\lambda_2}. \tag{11.9}$$

By Proposition 3.2 and (11.7), we have

$$\|\delta f_{02}\|_0 \leq \frac{C_2}{\lambda_2}. \tag{11.10}$$

Moreover, it's obvious that

$$2 \sum_{l \in Z^3} a_{2l}^2(t, x) k_2 \otimes k_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2\varphi^2(t, x) \end{pmatrix}.$$

Thus, by (11.2),

$$R_{01} + 2 \sum_{l \in Z^3} a_{2l}^2 k_2 \otimes k_2 = \delta R_{01}.$$

Finally, we have

$$R_{02} = \delta R_{01} + \delta R_{02}, \quad f_{02} = f_{01} + \delta f_{02}$$

and

$$\|R_{02}\|_0 \leq \frac{C_2}{\mu_2} + C_2 \frac{\mu_2}{\lambda_2} + C_1 \frac{\mu_1}{\lambda_1}, \quad \|f_{02}\|_0 \leq C_1 \frac{\mu_1}{\lambda_1} + \frac{C_2}{\lambda_2}.$$

Next, we claim $\|v_{02}\|_0 \geq 10M$. In fact,

$$v_{02} = v_{01} + w_2 = v_{01o} + w_{2o} + v_{01c} + w_{2c}.$$

By (11.1) and (11.6)

$$v_{01o} = 2 \sum_{l \in Z^3} a_{1l} \left(\sin(\lambda_1 2^{|l|} x_2), 0 \right)^T, \quad w_{2o} = -2 \sum_{l \in Z^3} a_{2l} \left(0, \sin \left(\lambda_2 2^{|l|} \left(x_1 - v_{01}^1 \left(\frac{l}{\mu_2} \right) t \right) \right) \right)^T,$$

where v_{01}^1 is the first component of v_{01} . If we set $(t, x_1, x_2) := (0, \frac{\pi}{2\lambda_2}, 0)$ and take $1 \ll \mu_2 \ll \lambda_2$, then

$$v_{01o}(t, x_1, x_2) = 0,$$

$$w_{2o}(t, x_1, x_2) = -2 \sum_{l \in Z^3} \varphi \left(0, \frac{\pi}{2\lambda_2}, 0 \right) \alpha_l \left(0, \frac{\mu_2 \pi}{2\lambda_2}, 0 \right) \left(0, \sin \left(\frac{2^{|l|} \pi}{2} \right) \right)^T = (0, -20M)^T.$$

Moreover, we can take $1 \ll \mu_1 \ll \lambda_1 \ll \mu_2 \ll \lambda_2$ such that

$$\|v_{01c}\|_0 + \|w_{2c}\|_0 \leq M.$$

Therefore, we conclude that

$$\|v_{02}\|_0 \geq 10M.$$

Finally, we set $(v_0, p_0, \theta_0, R_0, f_0) := (v_{02}, p_{02}, \theta_{02}, R_{02}, f_{02})$.

In conclusion, for any $M > 0$, $r > 0$, we can construct function $(v_0, p_0, \theta_0, R_0, f_0) \in C_c^\infty(Q_r)$ which solves Boussinesq-stress system (2.1) and satisfies the following estimates

$$\|R_0\|_0 \leq \frac{C_2}{\mu_2} + C_2 \frac{\mu_2}{\lambda_2} + C_1 \frac{\mu_1}{\lambda_1}, \quad \|f_0\|_0 \leq C_1 \frac{\mu_1}{\lambda_1} + \frac{C_2}{\lambda_2}, \quad \|v_0\|_0 \geq 10M, \quad \theta_0 \in \Theta(r, k_1).$$

Similarly, for $k_2 = (0, 1)^T$, we also can construct function $(v_0, p_0, \theta_0, R_0, f_0) \in C_c^\infty(Q_r)$ which solves Boussinesq-stress system (2.1) and satisfies the following estimates

$$\|R_0\|_0 \leq \frac{C_2}{\mu_2} + C_2 \frac{\mu_2}{\lambda_2} + C_1 \frac{\mu_1}{\lambda_1}, \quad \|f_0\|_0 \leq C_1 \frac{\mu_1}{\lambda_1} + \frac{C_2}{\lambda_2}, \quad \|v_0\|_0 \geq 10M, \quad \theta_0 \in \Theta(r, k_2).$$

11.2. Proof of Theorem 1.1.

Proof. From subsection 11.1, we know that for any $r > 0$, $k = (0, 1)^T$ or $(1, 0)^T$, there exists $(v_0, p_0, \theta_0, R_0, f_0) \in C_c^\infty(Q_r)$ which solves Boussinesq-stress system (2.1) and satisfies the following estimates:

$$\|R_0\|_0 \leq \frac{C_2}{\mu_2} + C_2 \frac{\mu_2}{\lambda_2} + C_1 \frac{\mu_1}{\lambda_1}, \quad \|f_0\|_0 \leq C_1 \frac{\mu_1}{\lambda_1} + \frac{C_2}{\lambda_2}, \quad \|v_0\|_0 \geq 10M, \quad \theta_0 \in \Theta(r, k).$$

Take $a, b \geq \frac{3}{2}$ such that $\frac{1}{a} \leq \min\{\frac{r}{2}, \frac{\varepsilon^2}{16M^2}\}$ and set $\delta_n := a^{-bn} : n = 0, 1, 2, \dots$. Then taking $\mu_1, \mu_2, \lambda_1, \lambda_2$ with $1 \ll \mu_1 \ll \lambda_1 \ll \mu_2 \ll \lambda_2$ such that

$$\|R_0\|_0 \leq \eta\delta_0, \quad \|f_0\|_0 \leq \eta\delta_0.$$

Applying Proposition 2.1 iteratively, we can construct

$$(v_n, p_n, \theta_n, R_n, f_n) \in C_c^\infty(Q_{r+\sum_{i=0}^n \delta_i}), \quad n = 1, 2, \dots$$

such that they solve system 2.1 and satisfy the following estimates

$$\|R_n\|_0 \leq \eta\delta_n, \tag{11.11}$$

$$\|f_n\|_0 \leq \eta\delta_n, \tag{11.12}$$

$$\|v_{n+1} - v_n\|_0 \leq M\sqrt{\delta_n}, \tag{11.13}$$

$$\|\theta_{n+1} - \theta_n\|_0 \leq M\sqrt{\delta_n}, \tag{11.14}$$

$$\|p_{n+1} - p_n\|_0 \leq M\delta_n, \tag{11.15}$$

$$\begin{aligned} \Lambda_{n+1} &:= \max\{1, \|R_{n+1}\|_{C_{t,x}^1}, \|f_{n+1}\|_{C_{t,x}^1}, \|v_{n+1}\|_{C_{t,x}^1}, \|\theta_{n+1}\|_{C_{t,x}^1}\} \\ &\leq A(\sqrt{\delta_n})^{\varepsilon^2+3\varepsilon+3} \left(\frac{\sqrt{\delta_n}}{\delta_{n+1}}\right)^{(1+\varepsilon)^2(2+\varepsilon)+(2+\varepsilon)^2} \Lambda_n^{(1+\varepsilon)^3}. \end{aligned} \tag{11.16}$$

In particular,

$$\|p_n\|_{C_{t,x}^1} \leq C_0, \quad \|\theta_{n+1}\|_{C_{t,x}^1} \leq A(\sqrt{\delta_n})^{2+\varepsilon} \left(\frac{\sqrt{\delta_n}}{\delta_{n+1}}\right)^{4+4\varepsilon+\varepsilon^2} \Lambda_n^{(1+\varepsilon)^2}. \tag{11.17}$$

It is obvious that $\sum_{i=0}^\infty \delta_i < r$. Thus, by (11.10)-(11.15), we know that $(v_n, p_n, \theta_n, R_n, f_n)$ are Cauchy sequence in $C_c(Q_{2r})$, therefore there exists

$$(v, p, \theta) \in C_c(Q_{2r})$$

such that

$$v_n \rightarrow v, \quad p_n \rightarrow p, \quad \theta_n \rightarrow \theta, \quad R_n \rightarrow 0, \quad f_n \rightarrow 0$$

in $C_c(Q_{2r})$.

By (11.13)

$$\|v - v_0\|_0 \leq M \sum_{n=0}^\infty \sqrt{\delta_n} < 4M.$$

Therefore

$$\|v\|_0 \geq 6M, \quad v \neq 0.$$

By (11.14)

$$\|\theta - \theta_0\|_0 \leq M \sum_{n=0}^\infty \sqrt{\delta_n} \leq \frac{4M}{\sqrt{a}} < \varepsilon.$$

Passing into the limit in (2.1), we conclude that v, p, θ solve (1.1) in the sense of distribution.

Next, we prove that the solution v, p, θ is Hölder continuous. We claim that for a suitable choice of a, b , there exist a constant $c > 1$ such that

$$\Lambda_n \leq a^{cb^n}.$$

We prove this claim by induction.

Indeed, for $n = 0$, it's obvious if we take $a \geq \Lambda_0 := \max\{1, \|R_0\|_0, \|f_0\|_0, \|v_0\|_0, \|\theta_0\|_0\}$. Assuming that we have proved $\Lambda_n \leq a^{cb^n}$, then

$$\Lambda_{n+1} \leq A(\sqrt{\delta_n})^{\varepsilon^2+3\varepsilon+3} \left(\frac{\sqrt{\delta_n}}{\delta_{n+1}} \right)^{(1+\varepsilon)^2(2+\varepsilon)+(2+\varepsilon)^2} \Lambda_n^{(1+\varepsilon)^3} \leq Aa^{-\frac{\varepsilon}{2}b^n} a^{cb^{n+1}} a^{((b-\frac{1}{2})d-\frac{\varepsilon^2+3\varepsilon+3}{2}+\frac{\varepsilon}{2}+c(1+\varepsilon)^3-cb)b^n}.$$

where $d = (1+\varepsilon)^2(2+\varepsilon) + (2+\varepsilon)^2$.

Take $b = \frac{3}{2}$ and $c = \frac{2d-(\varepsilon^2+2\varepsilon+3)}{3-2(1+\varepsilon)^3}$, we arrive at

$$\Lambda_{n+1} \leq Aa^{-\frac{\varepsilon}{2}} a^{cb^{n+1}}.$$

Then choosing $a \geq A^{\frac{2}{\varepsilon}}$, we have $\Lambda_{n+1} \leq a^{cb^{n+1}}$. Finally, we take

$$a := \max \left\{ A^{\frac{2}{\varepsilon}}, \Lambda_0, \frac{3}{2}, \frac{1}{\min\{\frac{r}{2}, \frac{\varepsilon^2}{16M^2}\}} \right\},$$

then a satisfies all the needed conditions.

Now we consider the approximate sequence v_n, p_n, θ_n . By (11.13), we have

$$\|v_{n+1} - v_n\|_0 \leq Ma^{-\frac{1}{2}b^n}.$$

Moreover, we have

$$\|v_{n+1} - v_n\|_{C_{t,x}^1} \leq \Lambda_n + \Lambda_{n+1} \leq 2a^{cb^{n+1}}.$$

Therefore, for any $\alpha \in (0, 1)$

$$\|v_{n+1} - v_n\|_{C_{t,x}^\alpha} \leq 2Ma^{(\alpha cb - \frac{(1-\alpha)}{2})b^n}.$$

If $\alpha < \frac{1}{1+2bc}$, then $\alpha cb - \frac{(1-\alpha)}{2} < 0$, thus v_n are Cauchy sequence in $C_{t,x}^\alpha$. Take the value of b, c , we know that $v \in C_{t,x}^\alpha$ for any $\alpha < \frac{3-2(1+\varepsilon)^3}{3-2(1+\varepsilon)^3+6d-3(\varepsilon^2+2\varepsilon+3)}$. When $\varepsilon \rightarrow 0$, we have $\frac{3-2(1+\varepsilon)^3}{3-2(1+\varepsilon)^3+6d-3(\varepsilon^2+2\varepsilon+3)} \rightarrow \frac{1}{28}$. By (11.15) and (11.17), we know that $p \in C_{t,x}^\beta$ for any $\beta \in (0, 1)$. By (11.14) and (11.17), we have

$$\|\theta_{n+1} - \theta_n\|_0 \leq Ma^{-\frac{1}{2}b^n}$$

and

$$\|\theta_{n+1} - \theta_n\|_{C_{t,x}^1} \leq 2a^{(3+4\varepsilon+\varepsilon^2+c(1+\varepsilon)^2)b^n}.$$

By interpolation, for any $\gamma \in (0, 1)$, we have

$$\|\theta_{n+1} - \theta_n\|_{C_{t,x}^\gamma} \leq 2Ma^{(\gamma(3+4\varepsilon+\varepsilon^2+c(1+\varepsilon)^2)-\frac{1-\gamma}{2})b^n}.$$

Take $\gamma < \frac{1}{1+2(3+4\varepsilon+\varepsilon^2+c(1+\varepsilon)^2)}$, then θ_n converge in C^γ , which implies that $\theta \in C_{t,x}^\gamma$ for any $\gamma < \frac{1}{1+2(3+4\varepsilon+\varepsilon^2+c(1+\varepsilon)^2)}$. When $\varepsilon \rightarrow 0$, we have $\frac{1}{1+2(3+4\varepsilon+\varepsilon^2+c(1+\varepsilon)^2)} \rightarrow \frac{1}{25}$. Thus, we complete our proof of theorem 1.1. \square

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